STUDY OF THE FLOW OF A RADIATING GAS BY A DIFFERENTIAL APPROXIMATION

by
Ping Cheng

A dissertation submitted to the Department of Aeronautics and Astronautics and the committee on the graduate division of Stanford University in partial fulfillment of the requirements for the degree of Doctor of Philosophy

January 1965

STUDY OF THE FLOW OF A RADIATING GAS BY A DIFFERENTIAL APPROXIMATION

A DISSERTATION

SUBMITTED TO THE DEPARTMENT OF AERONAUTICS AND ASTRONAUTICS

AND THE COMMITTEE ON THE GRADUATE DIVISION

OF STANFORD UNIVERSITY

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS

FOR THE DEGREE OF

DOCTOR OF PHILOSOPHY

by

Ping Cheng

January 1965

ACKNOWLEDGEMENTS

The author wishes to express his gratitude to Professor Walter G. Vincenti for his valuable discussion, encouragement, and inspiration during the course of study. He also wishes to thank Dr. B. S. Baldwin, Jr. for helpful discussion during the early phase of the work. Thanks are also due to Professor Joel H. Ferziger for helpful discussion on the spherical-harmonic method.

The work presented here was supported by the U.S. Air Force Office of Scientific Research under contract AF49(638)1280.

TABLE OF CONTENTS

LIST OF ILLUSTRATIONS

I. INTRODUCTION

The effects of thermal radiation in compressible flow have received considerable attention in recent years as a result of the increasing flight speeds connected with space flight. Little work has been done, however, in multi-dimensional flow problems with nonequilibrium heat transfer due to radiation. In fact, no solutions valid for the complete range of absorption coefficient have been obtained for such problems. The difficulty is due essentially to the fact that the governing equations are of a complicated integro-differential form.

The present work represents an attempt to understand various aspects of multi-dimensional radiating gas flow with no restriction as to the magnitude of the absorption coefficient. Chapters II and III provide background material that is needed for subsequent discussion. In particular, Chapter II reviews briefly the fundamentals of radiation transport. In this, emphasis is placed on the particle characteristics of radiation. This approach enables us to discuss the problems of radiation transport in a manner analogous to that used in kinetic theory of gases and in neutron-transport theory.

The basic integro-differential equations of radiation gasdynamics are reviewed in Chapter III. For one-dimensional problems, the integral term in the differential equations can be removed by the so-called "exponential approximation" suggested by Vincenti and Baldwin (1962).

As a result, analytical solutions valid for the full range of absorption coefficient are possible in the one-dimensional case. Since the exponential approximation is related to the spherical-harmonic approximation that we shall discuss at length in subsequent chapters, a brief treatment of this approximation is included.

Although the exponential approximation does lead to analytical solutions valid for the complete range of absorption coefficient in the case of one-dimensional flow, this technique is not applicable to multi-dimensional problems. We therefore look for other approximate methods that can be used in multi-dimensional flow. One such method, known as the moment method, has been discussed recently by Traugott (1963) and by the present author (1964). With this approximation, the exact radiation transport equation can be replaced by a set of its moment equations. This set of approximate transport equations, together with the gas-dynamic equations, constitute a determinate set of purely differential equations valid for the complete range of absorption coefficient. In Section 4.1, the same approximate transport equations for a multi-dimensional radiating gas are rederived as the first approximation in a systematic spherical-harmonic method similar to that used in astrophysics and neutron-transport theory.

In the approximate treatment a question arises as to how the boundary conditions on the radiation field are to be approximated in a manner consistent with the spherical-harmonic approach. To handle this question the so-called "Mark" and "Marshak" boundary conditions of neutron-transport theory are applied to the present problem. As a result, expressions for the approximate boundary condition consistent with the

spherical-harmonic method are given for a black surface in Section 4.2. With the spherical-harmonic approximation, the governing equations and the boundary conditions are in purely differential form. We therefore refer to the present formulation as the "differential approximation" to radiating gas flow.

The relation of the exponential approximation and the present differential approximation is discussed in Section 4.3. It is shown that the exponential approximation with suitable choice of constants in the exponential function is completely equivalent to the differential approximation when specialized to one-dimensional situations and employed with the Mark boundary condition.

Even though the governing equations in the present formulation are in purely differential form, they are nonlinear, and analytical solution is still difficult. We therefore examine the corresponding linearized equations in Chapter V. As a result, a single, linear, fifth-order partial differential equation in terms of the velocity potential, with coordinate system fixed in the undisturbed fluid, is obtained. The corresponding equation for a coordinate system fixed in a body moving with constant speed is found by means of a Galilean transformation. These two equations play the same role as the acoustic equation and the Prandtl-Glauert equation respectively in classical gasdynamics. Within the framework of the linearized theory, an expression for the boundary condition on the wall temperature is given. This expression shows that, except in the limiting case of an opaque gas, a temperature jump exists at the wall.

To illustrate the present formulation of radiating gas flow, several simple examples are considered in Chapter VI. In particular, the propagation of linearized one-dimensional waves, solved previously with the exponential approximation, is reconsidered on the basis of the differential approximation. It is shown that the results obtained from the two methods of approximation are essentially equivalent. Finally, with the desire to provide a simple analytical solution for multi-dimensional radiating flow, the two-dimensional problem of flow over a wavy wall is considered. So far as the author is aware, this is the first solution obtained for a multi-dimensional problem with no restriction on the magnitude of the absorption coefficient. The results show that, in general, two systems of waves are present in the flow field, a modified classical wave and a radiation-induced wave. The occurence of pressure drag at subsonic speeds and the smoothing of the transition from subsonic to supersonic speeds reflect the nonequilibrium character of the radiating flow.

II. FUNDAMENTALS OF RADIATION TRANSPORT

Most of the existing texts on thermal radiation were written by astrophysicists (for example, Chandrasekhar (1950), Kourganoff (1952), and Sobolev (1963)). The problems with which they deal are characterized by an absorbing, emitting, and scattering medium that is stationary, time-independent, and infinite in extent. In problems of radiative gasdynamics, aerodynamicists must often be concerned with time-dependent,

multi-dimensional situations in the presence of a radiating wall. In this chapter, we will review briefly the fundamentals of radiation-transport from the aerodynamicist's point of view. A useful discussion on various aspects of radiation-transport from this point of view can also be found in the works of Lighthill (1960), Vincenti and Baldwin (1962), and Goulard (1962).

2.1. Photons.

It was suggested by Planck (1900) that the emission and absorption of radiant energy by matter does not take place continuously, but in finite quanta of energy. Einstein (1905) went a step further to suggest that not only is the radiant energy emitted and absorbed in quanta of energy but that it travels through space in such quanta with the speed of light. The quanta of energy are commonly called photons. It is understood now that photons have the dual characteristics of both waves and particles. They show wave characteristics with regard to propagation, and particle characteristics during the interaction between radiation and matter. When the wave description is used, radiation is characterized by a wave length λ and a frequency ν connected by

$$\nu\lambda = c \quad , \tag{2.1}$$

where c is the speed of light. When the particle description is used, it is characterized by a momentum $\bar{P}$ and an energy E connected by

$$\bar{P} = \frac{E}{c}\bar{\Omega} \quad , \tag{2.2}$$

where $\bar{\Omega}$ is the unit vector in the direction in which the photon is moving. The energy E in Eq. (2.2) is given by the expression

$$E = h\nu \quad , \tag{2.3}$$

where h is Planck's constant.

2.2. The Distribution Function and the Radiation Intensity.

Since photons have the characteristics of particles, we may, as in the kinetic theory, use the distribution function to characterize the radiation field. The distribution function f_ν is defined as the number of photons "belonging to $d\nu\, d\Omega$" (i.e., photons in the frequency interval $d\nu$, moving within the element of solid angle $d\Omega$ centered about the unit vector $\bar{\Omega}$) per unit volume at time t.

Astrophysicists have found it convenient to use the radiation intensity to characterize the radiation field. The radiation intensity is defined as the amount of radiant energy carried by photons "belonging to $d\nu\, d\Omega$" that crosses a unit area per unit time. Consider an arbitrary area element dA with its orientation designated by a unit vector $\bar{n}$ normal to dA. (In the following, we shall follow the convention that the orientation of the surface element of a body is designated by the outward-drawn unit vector normal to dA.) During a time interval dt, those photons "belonging to $d\nu\, d\Omega$" that cross dA at the beginning of

the time interval have travelled a distance cdt. The trajectory of these photons has generated a cylinder with base dA and slant height cdt in the direction of $\bar{\Omega}$. Thus the number of such photons crossing dA during the time interval dt is

$$f_\nu \, c \, \bar{\Omega} \cdot \bar{n} \, dA \, dt \; . \tag{2.4}$$

It follows that the amount of radiant energy carried by such photons crossing dA during time interval dt is

$$h\nu \, f_\nu \, c \, \bar{\Omega} \cdot \bar{n} \, dA \, dt \; . \tag{2.5}$$

By definition, the radiation intensity I_ν is obtained by dividing expression (2.5) by the time interval dt and the area element perpendicular to $\bar{\Omega}$. Thus we have

$$I_\nu = \frac{h\nu \, f_\nu \, c \, \bar{\Omega} \cdot \bar{n} \, dAdt}{\bar{n} \cdot \bar{\Omega} \, dA \, dt} = h\nu \, c \, f_\nu \; . \tag{2.6}$$

From the definitions of the distribution function and the radiation intensity, it follows that these quantities, in general, are functions of frequency ν (or energy $E = h\nu$), position $\bar{r}$, direction $\bar{\Omega}$, and time t. To recognize this explicitly, we can write

$$\begin{aligned} f_\nu &= f_\nu(\bar{r}, \bar{\Omega}, t) \; , \\ I_\nu &= I_\nu(\bar{r}, \bar{\Omega}, t) \; . \end{aligned} \tag{2.7}$$

2.3. Radiant Energy Density, Space-Integrated Radiation Intensity, Radiant Heat-Flux Vector, and Radiant Pressure Tensor.

The radiant energy density is defined as the radiant energy per unit volume per unit frequency. It follows from the definition of the distribution function that the specific radiant energy density of photons moving in the direction of $\bar{\Omega}$ is $h\nu f_\nu \, d\Omega$. The radiant energy density u_{r_ν} is therefore given by

$$u_{r_\nu}(\bar{r}, t) = \int_\Omega h\nu f_\nu \, d\Omega \, , \tag{2.8}$$

where the suffix Ω signifies that the integration is carried out over the entire range of solid angle. With the help of Eq. (2.6), Eq. (2.8) can be expressed in terms of I_ν as

$$u_{r_\nu}(\bar{r}, t) = \frac{1}{c} \int_\Omega I_\nu(\bar{r}, \bar{\Omega}, t) \, d\Omega \, . \tag{2.9}$$

The space-integrated radiation intensity is defined by

$$I_{o_\nu}(\bar{r}, t) \equiv \int_\Omega I_\nu(\bar{r}, \bar{\Omega}, t) \, d\Omega \quad . \tag{2.10}$$

It follows from Eqs. (2.9) and (2.10) that the radiant energy density and the space-integrated radiation intensity are related by

$$u_{r_\nu}(\bar{r}, t) = \frac{I_{o_\nu}(\bar{r}, t)}{c} \, . \tag{2.11}$$

We are now in a position to find the radiant heat-flux vector and the radiant pressure tensor. To this end, let us suppose that associated with each photon, there is some property g, the magnitude of which depends on c. From expression (2.4), it follows that

$$g\, f_\nu\, c\, \bar{\Omega} \cdot \bar{n}\, dA\, dt \tag{2.12}$$

represents the amount of this property transported across dA during the time interval dt by photons "belonging to $d\nu\, d\Omega$". Thus

$$g\, f_\nu\, c\, \bar{\Omega} \cdot \bar{n}\, d\Omega\ , \tag{2.13}$$

is the amount of g per unit frequency, carried by photons moving within the solid angle $d\Omega$ about $\bar{\Omega}$, per unit area per unit time. The net flux of g across the surface element is obtained by integrating this expression over all directions. Thus we have

$$\int_\Omega g\, f_\nu\, c\, \bar{\Omega} \cdot \bar{n}\, d\Omega = (\bar{n} \cdot \int_\Omega g\, f_\nu\, c\, \bar{\Omega}\, d\Omega) = (\bar{n} \cdot \bar{\Psi})\ , \tag{2.14}$$

where the vector

$$\bar{\Psi} \equiv \int_\Omega g\, f_\nu\, c\, \bar{\Omega}\, d\Omega\ , \tag{2.15}$$

is called the flux vector associated with the property g. This vector has the physical significance that the component of the vector in the direction $\bar{n}$ is the flux of the associated physical property across a surface normal to $\bar{n}$.

Consider first the transfer of energy flux. If we let $g = h\nu$, Eq. (2.15) becomes

$$\bar{\Psi} = \int_{\Omega} h\nu \, c \, f_{\nu} \, \bar{\Omega} \, d\Omega = \int_{\Omega} I_{\nu} \, \bar{\Omega} \, d\Omega \equiv \bar{q}_{\nu}(\bar{r}, t), \tag{2.16}$$

where $\bar{q}_{\nu}$ is the radiant heat-flux vector. The component of this vector in the direction parallel to the coordinate x_i is

$$q_{\nu_i}(\bar{r},t) = \bar{n}_i \cdot \bar{q}_{\nu} = \int_{\Omega} \bar{n}_i \cdot \bar{\Omega} \, I_{\nu}(\bar{r}, \bar{\Omega}, t) \, d\Omega = \int_{\Omega} \ell_i \, I_{\nu}(\bar{r}, \bar{\Omega}, t) \, d\Omega, \tag{2.17}$$

where $\bar{n}_i$ is the unit vector in the direction x_i and ℓ_i is the direction cosine of the unit vector $\bar{\Omega}$ with respect to x_i. For the coordinate system in Fig. 1, we have $d\Omega = \sin\theta \, d\theta \, d\phi$, and $\bar{\Omega} = \ell_1\bar{n}_1 + \ell_2\bar{n}_2 + \ell_3\bar{n}_3$, where

$$\ell_1 = \bar{n}_1 \cdot \bar{\Omega} = \sin\theta \sin\phi, \quad \ell_2 = \bar{n}_2 \cdot \bar{\Omega} = \cos\theta \ , \qquad \ell_3 = \bar{n}_3 \cdot \bar{\Omega} = \sin\theta \cos\phi \ . \tag{2.18}$$

More generally, the component of $\bar{q}_{\nu}$ in the direction of $\bar{n}$ is given by

$$q_{\nu_n}(\bar{r}, t) = \bar{n} \cdot \bar{q}_{\nu} = \int_{\Omega} \bar{n} \cdot \bar{\Omega} \, I_{\nu}(\bar{r}, \bar{\Omega}, t) \, d\Omega. \tag{2.19}$$

Consider next the momentum flux associated with the transport of photons. If we let $g = \bar{P} \cdot \bar{n}_j = (h\nu \bar{\Omega} \cdot \bar{n}_j)/c$, then

$$\bar{Y} = \int_\Omega h\nu \bar{\Omega} \cdot \bar{n}_j f_\nu \bar{\Omega} \, d\Omega = \frac{1}{c} \int_\Omega \bar{\Omega} \cdot \bar{n}_j I_\nu \bar{\Omega} \, d\Omega \quad , \qquad (2.20)$$

is the flux vector associated with the transport of the $\bar{n}_j$-component of momentum. Since pressure $p_{\nu_{ij}}$ is defined as the net rate of transfer of the $\bar{n}_j$-component of momentum per unit area normal to the unit vector $\bar{n}_i$, we have

$$p_{\nu_{ij}}(\bar{r},t) = \frac{\bar{n}_i}{c} \cdot \int_\Omega \bar{\Omega} \cdot \bar{n}_j I_\nu \bar{\Omega} \, d\Omega$$

$$= \frac{1}{c} \int_\Omega \bar{\Omega} \cdot \bar{n}_i \bar{\Omega} \cdot \bar{n}_j I_\nu \, d\Omega = \frac{1}{c} \int_\Omega \ell_i \ell_j I_\nu \, d\Omega \quad , \qquad (2.21)$$

which is a symmetric second-order tensor.

2.4. Splitting of the Space-Integrated Radiation Intensity and the Radiant Heat Flux.

It is sometimes convenient to distinguish quantities associated with photons moving in certain directions by superscripts + and -. Imagine a transparent plane with its orientation denoted by $\bar{n}$. The notations f_ν^+ and I_ν^+ are respectively the distribution function and the radiation intensity associated with those photons moving in the direction of the unit vector $\bar{\Omega}$ where $\bar{\Omega} \cdot \bar{n} > 0$. The space-integrated radiation intensity and the radiant heat flux associated with these photons are given respectively by

$$I_{o_\nu}^{+}(\bar{r},t) \equiv \int_{\bar{\Omega}\cdot\bar{n}>0} I_\nu^{+}(\bar{r}, \bar{\Omega}, t)\, d\Omega, \qquad (2.22)$$

and

$$q_{\nu_n}^{+}(\bar{r},t) \equiv \int_{\bar{\Omega}\cdot\bar{n}>0} \bar{\Omega}\cdot\bar{n}\; I_\nu^{+}(\bar{r},\bar{\Omega},t)\, d\Omega, \qquad (2.23)$$

where the suffix $\bar{\Omega} \cdot \bar{n} > 0$ signifies that the integration is extended only over the hemisphere where $\bar{\Omega} \cdot \bar{n} > 0$.

Similarly, if f_ν^- and I_ν^- are respectively the distribution function and the radiation intensity associated with photons moving in the direction of the unit vector $\bar{\Omega}$ where $\bar{\Omega} \cdot \bar{n} < 0$, the space-integrated radiation intensity and the radiant heat flux associated with these photons are

$$I_{o_\nu}^{-}(\bar{r}, t) \equiv \int_{\bar{\Omega}\cdot\bar{n}<0} I_\nu^{-}(\bar{r}, \bar{\Omega}, t)\, d\Omega , \qquad (2.24)$$

and

$$q_{\nu_n}^{-}(\bar{r}, t) \equiv - \int_{\bar{\Omega}\cdot\bar{n}<0} \bar{\Omega}\cdot\bar{n}\; I_\nu^{-}(\bar{r}, \bar{\Omega}, t)\, d\Omega, \qquad (2.25)$$

where the suffix $\bar{\Omega} \cdot \bar{n} < 0$ signifies that the integration is extended only over the hemisphere where $\bar{\Omega} \cdot \bar{n} < 0$. Note that $I_{o_\nu}^{+}$, $q_{\nu_n}^{+}$, $I_{o_\nu}^{-}$ and $q_{\nu_n}^{-}$ given respectively by Eqs. (2.22) through (2.25) are all positive quantities.

It follows from Eqs. (2.10), (2.19), (2.22) through (2.25) that

$$I_{o_\nu}(\bar{r},t) = I^{+}_{o_\nu}(\bar{r}, t) + I^{-}_{o_\nu}(\bar{r}, t) \,, \qquad (2.26)$$

and

$$q_{\nu_n}(\bar{r},t) = q^{+}_{\nu_n}(\bar{r}, t) - q^{-}_{\nu_n}(\bar{r}, t) \,. \qquad (2.27)$$

2.5. Emission Coefficient, Absorption Coefficient, Scattering Coefficient, and the Scattering Function.

Radiation interacts with matter through emission, absorption, and scattering. The absorption of photons by a molecule raises its rotational, vibrational, or electronic energy level. Conversely, by emitting photons, a molecule lowers its energy level. The number of photons "belonging to $d\nu\, d\Omega$" emitted in the volume dV during time dt is

$$\left(\frac{e_\nu}{h\nu}\right) d\nu\, d\Omega\, dV\, dt \,, \qquad (2.28)$$

where e_ν is the emission coefficient which is a function of position, angular direction, frequency, and time.

It should be noted that photons do not collide with each other and that they all move with the same speed, namely, the speed of light.

Therefore, in considering the collisions between photons and molecules, we may assume that the molecules are at rest. As a result of collisions, photons may be absorbed or scattered. The decrease in photons "belonging to $d\nu\, d\Omega$" in volume dV during time dt, as a result of absorption and scattering can be considered to be proportional to the path length ds and the number of such photons in dV. If α_ν and η_ν are the absorption and scattering coefficients respectively, the quantities

$$\alpha_\nu\, f_\nu(\bar{r}, \bar{\Omega}, t)\, d\nu\, d\Omega\, dV\, ds \tag{2.29}$$

and

$$\eta_\nu\, f_\nu(\bar{r}, \bar{\Omega}, t)\, d\nu\, d\Omega\, dV\, ds \tag{2.30}$$

represent the decrease in such photons as a result of absorption and scattering. Both α_ν and η_ν are functions of frequency, position, and time. The reciprocal of the absorption coefficient is called the mean free path of photons.

It follows from Eq. (2.30) that the number of photons "belonging to $d\nu\, d\Omega$" in volume dV during time dt, scattering from the direction $\bar{\Omega}'$ to other directions is

$$\eta_\nu(\bar{r},t)\, f_\nu(\bar{r}, \bar{\Omega}', t)\, d\nu\, d\Omega'\, dV\, ds\ . \tag{2.31}$$

Of this number, a certain fraction $\kappa_\nu(\bar{r}, \bar{\Omega}' \to \bar{\Omega}, t)\, d\Omega$ will emerge from these collisions moving in directions that lie in the element of solid angle $d\Omega$ about $\bar{\Omega}$. Here κ_ν is called the scattering function. Thus the expression

$$\kappa_\nu(\bar{r}, \bar{\Omega}' \to \bar{\Omega}, t)\ \eta_\nu(\bar{r}, t)\ f_\nu(\bar{r}, \bar{\Omega}', t)\ d\nu\ d\Omega'\ dV\ ds\ d\Omega, \quad (2.32)$$

represents the number of photons initially directed along $\bar{\Omega}'$ which are scattered into direction about $\bar{\Omega}$ while in the volume dV. The integral of this expression, namely,

$$\int_{\Omega'} \kappa_\nu(\bar{r}, \bar{\Omega}' \to \bar{\Omega}, t)\ \eta_\nu(\bar{r}, t)\ f_\nu(\bar{r}, \bar{\Omega}', t)\ d\nu\ d\Omega'\ dV\ ds\ d\Omega, \quad (2.33)$$

gives the total number of such contributions from all directions. The scattering function is normalized such that

$$\int_{\Omega'} \kappa_\nu(\bar{r}, \bar{\Omega}' \to \bar{\Omega}, t)\ d\Omega' = 1 . \quad (2.34)$$

2.6. Radiation-Transport Equation.

The equation of radiation transport is an equation of conservation of photons, or radiant energy. Let us consider a packet of photons "belonging to $d\nu\ d\Omega$" occupying an arbitrary volume element dV at time t. After a time interval dt, the number of photons in the packet will decrease on account of absorption by matter and scattering out of the packet as the result of collisions; it will increase on account of emission and scattering into the packet. The net rate of increase of photons in the packet is

$$\frac{d}{dt}\left[f_\nu \, d\nu \, d\Omega \, dV\right] = \left[\frac{\partial f_\nu}{\partial t} + c\,\bar{\Omega}\cdot \text{grad } f\right] d\nu \, d\Omega \, dV. \tag{2.35}$$

From Eq. (2.28), the rate of increase of such photons in the packet as the result of emission in dV is

$$\frac{e_\nu}{h\nu} \, d\nu \, d\Omega \, dV \quad . \tag{2.36}$$

It follows from Eq. (2.33) that the rate of increase of such photons in dV as a result of collisions which scatter photons from other directions $\bar{\Omega}'$ to the direction $\bar{\Omega}$ is

$$\eta_\nu(\bar{r},t) \int_{\Omega'} \kappa_\nu(\bar{r}, \bar{\Omega}' \to \bar{\Omega}, t)\, f_\nu(\bar{r}, \bar{\Omega}', t)\, d\nu \, d\Omega' \, dV \, ds \, d\Omega/dt \; . \tag{2.37}$$

As given by Eqs. (2.29) and (2.30), the rate of decrease of such photons as a result of absorption and scattering is

$$(\alpha_\nu + \eta_\nu)\, f_\nu(\bar{r}, \bar{\Omega}, t)\, d\nu \, d\Omega \, dV \, ds/dt \quad . \tag{2.38}$$

From conservation requirements, collecting the results of Eqs. (2.35) through Eq. (2.38), and using the relation $ds = cdt$, we have finally

$$\frac{\partial f_\nu}{\partial t} + c\,\bar{\Omega}\cdot \text{grad } f_\nu \tag{2.39}$$

$$= \frac{e_\nu}{h\nu} - (\alpha_\nu + \eta_\nu)\, f_\nu c + c\eta_\nu \int_{\Omega'} \kappa_\nu(\bar{r}, \bar{\Omega}' \to \bar{\Omega}, t)\, f_\nu(\bar{r}, \bar{\Omega}', t)\, d\Omega' .$$

Eq. (2.39) is the radiation-transport equation in terms of the distribution function. For a purely scattering medium, this equation reduces to a form similar to the Boltzmann equation in kinetic theory. The only difference lies in the scattering term. In kinetic theory, the integral term is nonlinear and is integrated with respect to velocity space, whereas in the radiation-transport equation, the scattering term is linear and is integrated with respect to solid angle. This is due to the fact that in the kinetic theory collision is between like molecules with different speeds, whereas in radiation-transport theory, collision is between photons with constant speed and molecules at rest. With certain changes in notation, Eq. (2.39) is also the governing equation for neutron transport with a constant-cross-section approximation.

In view of Eq. (2.6), Eq. (2.39) can be rewritten in terms of radiation intensity as

$$\frac{1}{c}\frac{\partial I_\nu}{\partial t} + \bar{\Omega} \cdot \text{grad } I_\nu \tag{2.40}$$

$$= e_\nu - (\alpha_\nu + \eta_\nu)\, I_\nu + \eta_\nu \int_{\Omega'} \kappa_\nu(\bar{r}, \bar{\Omega}' \to \bar{\Omega}, t)\, I_\nu(\bar{r}, \bar{\Omega}', t)\, d\Omega' .$$

The second term on the left-hand side of Eq. (2.40) can be written in a number of different forms. For example, we can write

$$\bar{\Omega} \cdot \text{grad } I_\nu = \ell_j \frac{\partial I_\nu}{\partial x_j} = \text{div}(\bar{\Omega}\, I_\nu) = \frac{\partial I_\nu}{\partial s}, \tag{2.41}$$

where s is the distance in the direction of the unit vector $\bar{\Omega}$.

2.7. Frequency-Integrated Quantities.

In the foregoing discussion, we have used the subscript ν to denote monochromatic quantities. To each monochromatic quantity Q_ν, we shall now associate a corresponding frequency-integrated quantity Q defined by

$$Q \equiv \int_0^\infty Q_\nu \, d\nu \quad . \tag{2.42}$$

Thus the expressions

$$I_o(\bar{r},t) \equiv \int_0^\infty I_{o_\nu} \, d\nu = \int_0^\infty \int_\Omega I_\nu \, d\Omega \, d\nu = \int_\Omega I(\bar{r},\bar{\Omega},t) \, d\Omega, \tag{2.43a}$$

$$u_r(\bar{r},t) \equiv \int_0^\infty u_{r_\nu} \, d\nu = \frac{1}{c} \int_0^\infty \int_\Omega I_\nu \, d\Omega \, d\nu = \frac{1}{c} \int_\Omega I(\bar{r},\bar{\Omega},t) \, d\Omega, \tag{2.43b}$$

$$q_i(\bar{r},t) \equiv \int_0^\infty q_{\nu_i} \, d\nu = \int_0^\infty \int_\Omega \ell_i \, I_\nu \, d\Omega \, d\nu = \int_\Omega \ell_i \, I(\bar{r}, \bar{\Omega},t) \, d\Omega, \tag{2.43c}$$

$$p_{ij}(\bar{r},t) \equiv \int_0^\infty p_{\nu_{ij}} \, d\nu = \frac{1}{c} \int_0^\infty \int_\Omega \ell_i \ell_j \, I_\nu \, d\Omega \, d\nu$$

$$= \frac{1}{c} \int_\Omega \ell_i \ell_j \, I(\bar{r}, \bar{\Omega}, t) \, d\Omega, \tag{2.43d}$$

are respectively the frequency-integrated radiation intensity, radiant energy density, radiant heat-flux and radiant pressure.

2.8. Complete Thermodynamic Equilibrium and Local Thermodynamic Equilibrium.

A system that is simultaneously in mechanical, thermal, and chemical equilibrium is said to be in thermodynamic equilibrium. Thus for a system to be in thermodynamic equilibrium, the temperature and pressure must be uniform throughout the system and no net chemical change can take place within the system. If the system is shielded from external radiation, it then contains uniform density of photons moving randomly in all directions, and the radiation field is homogenous everywhere.

If we denote quantities in complete thermodynamic equilibrium by superscript *, Eq. (2.40) gives, in view of Eq. (2.34),

$$\frac{e_\nu}{\alpha_\nu} = I_\nu^* \quad . \tag{2.44}$$

It is shown in quantum theory that if the system is in complete thermodynamic equilibrium, the equilibrium radiation intensity is given by the Planck function

$$I_\nu^* = \frac{2h\nu^3}{c^2} \frac{1}{\exp(\frac{h\nu}{KT}) - 1} \, , \tag{2.45}$$

where K is the Boltzmann constant and T is the temperature of the matter as well as of the radiation. It follows from Eqs. (2.44) and (2.45) that the emission and absorption coefficients are related to the Planck function as

$$\frac{e_\nu}{\alpha_\nu} = \frac{2h\nu^3}{c^2} \frac{1}{\exp(\frac{h\nu}{KT}) - 1} \quad . \tag{2.46a}$$

In astrophysics, the function B_ν is defined as

$$B_\nu \equiv \frac{e_\nu}{\alpha_\nu} = \frac{2h\nu^3}{c^2} \frac{1}{\exp(\frac{h\nu}{KT}) - 1} \quad . \tag{2.46b}$$

For a radiation field in equilibrium, Eqs. (2.9), (2.10), (2.17) and (2.21) yield, with the aid of Eqs. (2.18),

$$u^*_{r_\nu} = \frac{4\pi B_\nu}{c} = \frac{4\pi}{c}\left[\frac{2h\nu^3}{c^2} \frac{1}{\exp(\frac{h\nu}{KT}) - 1}\right] , \tag{2.47a}$$

$$I^*_{o_\nu} = 4\pi B_\nu , \tag{2.47b}$$

$$q^*_{\nu_i} = 0 \quad , \tag{2.47c}$$

$$p^*_{\nu_{ij}} = (\frac{4\pi}{3c})\, B_\nu\, \delta_{ij} \quad , \tag{2.47d}$$

where δ_{ij} is the Kronecker delta, and T is the temperature of the gas. It follows from Eqs. (2.47a) and (2.47d) that for a radiation field in equilibrium, the pressure tensor and the energy density are related by

$$p^*_{\nu_{ij}} = \frac{1}{3}\, u^*_{r_\nu}\, \delta_{ij} \quad . \tag{2.48}$$

Eqs. (2.22) through (2.25) when specialized to radiative equilibrium become

$$I^{+*}_{o_\nu} = I^{-*}_{o_\nu} = 2\pi\ B_\nu\ , \qquad (2.49a)$$

and

$$q^{+*}_{\nu_n} = q^{-*}_{\nu_n} = \pi B_\nu\ . \qquad (2.49b)$$

The frequency-integrated Planck function is given by

$$B = \int_0^\infty B_\nu\ d\nu = \int_0^\infty \frac{2h\nu^3}{c^2}\ \frac{d\nu}{\exp(\frac{h\nu}{KT}) - 1} = \frac{\sigma T^4}{\pi} = I^*\ , \qquad (2.50a)$$

where $\sigma \equiv (2\pi^5\ K^4/15h^3c^2)$ is the Stefan-Boltzmann constant. If we integrate Eqs. (2.47) through (2.49) over the complete range of frequency, we have, in view of Eq. (2.50a)

$$u^*_r = \frac{4\sigma T^4}{c}\ , \qquad (2.50b)$$

$$I^*_o = 4\sigma T^4\ , \qquad (2.50c)$$

$$q^*_{\nu_n} = 0\ , \qquad (2.50d)$$

$$p^*_{ij} = \frac{1}{3}\ u^*_r\ \delta_{ij} = \frac{4\sigma T^4}{3c}\ \delta_{ij}\ , \qquad (2.50e)$$

$$I^{+*}_o = I^{-*}_o = \frac{\sigma T^4}{\pi}\ , \qquad (2.50f)$$

$$q^{+*}_n = q^{-*}_n = \sigma T^4\ . \qquad (2.50g)$$

In the subsequent discussion, we shall define a black surface as the surface whose I^+, I_o^+, and q_n^+ are given respectively by Eqs. (2.50a), (2.50f), and (2.50g).

The concept of local thermodynamic equilibrium is useful when a system is out of equilibrium. We will speak of local thermodynamic equilibrium if at every point of the matter in question a definite temperature can be assigned. It should be noted that Eqs. (2.46) holds also when local thermodynamic equilibrium is assumed.

With the assumption of local thermodynamic equilibrium, Eq. (2.40) becomes, with the aid of Eq. (2.46b),

$$\frac{1}{c}\frac{\partial I_\nu}{\partial t} + \bar{\Omega} \cdot \text{grad } I_\nu \tag{2.51}$$

$$= \alpha_\nu B_\nu - (\alpha_\nu + \eta_\nu) I_\nu + \eta_\nu \int_{\Omega'} \kappa_\nu(\bar{r}, \bar{\Omega}' \rightarrow \bar{\Omega}, t)\, I_\nu(\bar{r}, \bar{\Omega}', t)\, d\Omega' \ .$$

2.9. Grey-Gas Approximation, Asymptotic Situations.

For most engineering problems, the effects of scattering can be neglected. In this case, the integro-differential equation (2.51) reduces to the purely differential equation

$$\frac{1}{c}\frac{\partial I_\nu}{\partial t} + \frac{\partial I_\nu}{\partial s} = -\alpha_\nu (I_\nu - B_\nu) \ . \tag{2.52}$$

Integrating Eq. (2.52) with respect to the entire range of frequency, we have

$$\frac{1}{c}\frac{\partial I}{\partial t} + \frac{\partial I}{\partial s} = -\int_0^\infty \alpha_\nu (I_\nu - B_\nu)\, d\nu \quad . \qquad (2.53)$$

The grey-gas approximation consists in replacing the absorption coefficient α_ν by a constant value α independent of frequency ν. With this simplification, Eq. (2.53) becomes

$$\frac{1}{c}\frac{\partial I}{\partial t} + \frac{\partial I}{\partial s} = -\alpha(I-B) \quad , \qquad (2.54)$$

where B is the integrated Planck function given by the expression (2.50a). Owing to the very large mangitude of c, the unsteady term in the radiation-transport equation is always much smaller than other terms for aerodynamic applications and can therefore be neglected even for time-dependent situations. If the unsteady term is neglected, Eq. (2.54) becomes

$$\frac{\partial I}{\partial s} = -\alpha(I-B) \quad . \qquad (2.55)$$

It can be shown that the general solution to this equation is

$$I(s) = e^{-\int \alpha(s) ds} \int \alpha(s') B(s') e^{\int \alpha(s)\, ds}\, ds' + C\, e^{-\int \alpha(s)\, ds} \quad , \qquad (2.56)$$

where C is to be determined from the boundary conditions. If we define the optical thickness as

$$\tau = \int_0^s \alpha(s)\, ds \quad , \tag{2.57}$$

and let the boundary value of I at τ^* be $I(\tau^*)$, Eq. (2.56) becomes

$$I(\tau) = \int_{\tau^*}^{\tau} B(\tau') \exp[-(\tau-\tau')]d\tau' + I(\tau^*) \exp[-(\tau-\tau^*)] \ . \tag{2.58}$$

Consider now the asymptotic expansion of Eq. (2.58). To this end, we shall speak of an optically thin or thick gas if the optical thickness given by Eq. (2.57) is much greater than or smaller than unity. Let us define the dimensionless quantities $\tilde{s} = s/\ell$ and $\tilde{\alpha} = \alpha/\alpha_o$, where ℓ is a characteristic length of the problem and α_o is a reference value of α. Equation (2.57) in terms of these dimensionless quantities becomes

$$\tau \equiv \alpha_o \ell \int_0^{\tilde{s}} \tilde{\alpha}\, d\tilde{s} \quad . \tag{2.59}$$

Since the dimensionless quantities are of order unity, the integral in Eq. (2.59) is of order one. It follows from Eq. (2.59) that the alternative definition of optically thin or thick gas depends whether $\alpha_o \ell$ is much greater than or smaller than unity.

For an optically thin gas, the magnitude of the quantities in square brackets in Eq. (2.58) is much less than unity. We may therefore expand the first exponential function in a power series in terms of $(\tau-\tau')$. For most engineering problems, the temperature of the wall is low compared with that of the gas and the last term in

Eq. (2.58) can therefore be neglected. If we retain only the first term of the series, Eq. (2.58) then becomes

$$I = \int_{\tau^*}^{\tau} B(\tau')\, d\tau' \quad . \tag{2.60}$$

It follows that

$$I \ll B \quad . \tag{2.61}$$

In view of Eqs. (2.41) and (2.50a), the radiation-transport equation (2.55) in this case can then be written as

$$\ell_j \frac{\partial I}{\partial x_j} = \frac{\alpha \sigma T^4}{\pi} \; . \tag{2.62}$$

That is, emission is a first-order effect in terms of the absorption coefficient, whereas self-absorbing is of second order. Thus an optically thin gas is non-absorbing. Equation (2.62) can be obtained alternatively by proceeding directly from the differential equation (2.55) as follows. We first expand the radiation intensity in a power series in α, namely, $I = \sum_{n=1}^{\infty} \alpha^n I^{(n)}$. Substituting this series into Eq. (2.55), collecting like powers of α, and setting the coefficient of α equal to zero, we then obtain an equation which is identical to Eq. (2.62).

If Eq. (2.62) is integrated over the entire range of solid angle, we have, with the help of Eq. (2.43c)

$$\frac{\partial q_j}{\partial x_j} = 4\alpha\sigma T^4 , \tag{2.63}$$

which is the thin-gas approximation.

For an optically thick gas, we assume that the function $B(\tau')$ in Eq. (2.58) may be expanded in a Taylor's series about $\tau' = \tau$. Furthermore, the last term in Eq. (2.58) is taken to be small and the lower limit of integration is replaced by $-\infty$. Thus Eq. (2.58) becomes

$$\begin{aligned} I(\tau) &= \int_{-\infty}^{\tau} d\tau' \, e^{-(\tau-\tau')} \left[B(\tau) + (\tau'-\tau) \frac{dB(\tau)}{d\tau} + \frac{(\tau'-\tau)^2}{2} \frac{d^2B(\tau)}{d\tau^2} + \cdots \right] \\ &= B(\tau) - \frac{dB(\tau)}{d\tau} + \frac{d^2B(\tau)}{d\tau^2} - \cdots \quad . \end{aligned} \tag{2.64}$$

Alternatively, Eq. (2.64) can be obtained from the differential equation (2.55) by first expanding the radiation intensity as

$$I(s) = \sum_{n=0}^{\infty} \left(\frac{1}{\alpha}\right)^n I^{(n)}(s) \quad . \tag{2.65a}$$

Substituting this series into Eq. (2.55), collecting like powers of α, and setting the coefficient of $(1/\alpha)^n$ equal to zero, we have

$$\begin{aligned} I^{(0)} &= B \ , \\ I^{(1)} &= -\frac{dI^{(0)}}{ds} = -\frac{dB}{ds} \ , \\ I^{(2)} &= -\frac{dI^{(1)}}{ds} = \frac{d^2B}{ds^2} \ , \\ &\vdots \end{aligned} \tag{2.65b}$$

Substituting Eqs. (2.65b) into Eq. (2.65a), we obtain an equation identical to Eq. (2.64).

If we retain the first three terms in Eq. (2.64), and substitute the resulting equation into Eq. (2.43c), we have

$$q_1(\bar{r}) = -\frac{16\sigma T^3}{3\alpha}\frac{\partial T}{\partial x_1} \quad . \qquad (2.66)$$

The radiant heat-flux given by Eq. (2.66) is called the thick-gas (or Rosseland) approximation. It is in a form similiar to that for heat conduction with a variable thermal conductivity given by $k = (16\sigma T^3/3\alpha)$.

2.10 Formal Solution of the One-Dimensional Radiation-Transport Equation.

We shall now restrict ourselves to the most simple class of problems in which the flow variables depend on only a single spatial coordinate, say x_2. Let us consider a semi-infinite extent of gas on one side of a planar solid surface perpendicular to the coordinate x_2 and located at $x_2 = 0$. If we define

$$\eta(x_2) = \int_0^{x_2} \alpha(x_2)\, dx_2 \ , \qquad (2.67a)$$

it follows that τ and η are related by

$$\tau = \frac{\eta(x_2) - \eta(x_2')}{\mu} = \frac{\eta-\eta'}{\mu} \ , \qquad (2.67b)$$

where $\mu \equiv \ell_2 \equiv \cos\theta$.

If we assume that the surface is black and has a temperature T_w, the value of I^+ at the wall is given by Eq. (2.50a) as

$$I^+(0) = \frac{\sigma T_w^4}{\pi} . \tag{2.68}$$

With the aid of relations (2.67) and (2.68), Eq. (2.58) can then be written as

$$I^+(\eta, \mu) = \frac{\sigma}{\pi} \int_0^{\eta} T^4(\eta') \exp\left[- \frac{(\eta-\eta')}{\mu}\right] \frac{d\eta'}{\mu} + \frac{\sigma T_w^4}{\pi} \exp\left[- \frac{\eta}{\mu}\right] , \tag{2.69a}$$

which is the radiation intensity directed away from the wall. Similarly, if no wall exists at infinity, the radiation intensity directed toward the wall as given by Eq. (2.58) is

$$I^-(\eta, \mu) = \frac{\sigma}{\pi} \int_{\eta}^{\infty} T^4(\eta') \exp\left[- \frac{(\eta'-\eta)}{\mu}\right] \frac{d\eta'}{\mu} . \tag{2.69b}$$

The space-integrated radiation intensity and the radiant heat-flux can be found by substituting Eqs. (2.69) into Eqs. (2.43a) and (2.43c). This leads to

$$I_0(\eta) = 2\sigma \int_0^{\eta} T^4(\eta') E_1(\eta-\eta') d\eta' + 2\sigma \int_{\eta}^{\infty} T^4(\eta')E_1(\eta'-\eta) d\eta' + 2\sigma T_w^4 E_2(\eta), \tag{2.70}$$

and

$$q(\eta) = 2\sigma \int_0^{\eta} T^4(\eta') E_2(\eta-\eta') d\eta' - 2\sigma \int_{\eta}^{\infty} T^4(\eta') E_2(\eta'-\eta) d\eta' + 2\sigma T_w^4 E_3(\eta) , \tag{2.71}$$

where the function $E_n(z)$ is defined by

$$E_n(z) \equiv \int_0^1 e^{-z/\mu} \mu^{n-2} d\mu . \qquad (2.72)$$

This function has the properties

$$E_n(z) = - E_n(-z) \quad , \qquad (2.73a)$$

$$E_n(z) = - \frac{d}{dz} E_{n+1}(z) \; , \qquad (2.73b)$$

$$E_n(0) = \frac{1}{n-1}, \qquad \text{if } n = 2, 3, 4, \ldots \qquad (2.73c)$$

$$E_n(\infty) = 0 \quad . \qquad (2.73d)$$

Equations (2.70) and (2.71) are the solutions obtained by Goulard (1960) for one-dimensional problems with a solid black boundary.

III. BASIC GAS-DYNAMIC EQUATIONS AND THE EXPONENTIAL APPROXIMATION

In this chapter, we shall discuss the fundamental equations of radiation gas-dynamics under the following assumptions:

1. The gas is in translational, vibrational, and chemical equilibrium. That is, the effects of viscosity, thermal conductivity, vibration, and dissociation are neglected.

2. The gas is perfect.

3. The gas is in local thermodynamic equilibrium.

4. The contributions of radiation pressur^ and radiation energy are neglected.

5. The effects of scattering are ne^ligible.

6. The absorption coefficient is independent of frequency, i.e., the grey-gas approximation can be applied.

Assumptions 1 and 2 are introduced to isolate the influence of radiation for subsequent discussion. In reality, for the temperature range in which the effects of radiation are important, the gas is nonperfect and other nonequilibrium processes occur as well. Assumptions 3, 4, and 5 are approximately satisfied for most engineering applications (see Lighthill (1960)). Assumption 6 is introduced to make the equations tractable.

3.1 Basic Equations.

A radiating gas can be considered as a mixture of gas molecules and photons. Since photons have zero rest mass, however, the continuity equation for the radiating gas is the same as in the classical case. Thus, we have

$$\frac{D\rho}{Dt} + \rho \frac{\partial u_j}{\partial x_j} = 0 , \qquad (3.1)$$

where ρ is the mass density and u_i $(i = 1, 2, 3)$ are the velocity components. The derivative $D/Dt \equiv \partial/\partial t + u_j \, \partial/\partial x_j$ is the substantial derivative, where the repeated dummy subscripts denote the summation convention.

If radiation pressure, viscosity, and body force are neglected the momentum equations are

$$\frac{Du_i}{Dt} + \frac{1}{\rho}\frac{\partial p}{\partial x_i} = 0 \qquad (i = 1, 2, 3), \qquad (3.2)$$

where p is the pressure of the gas.

The energy equation can be obtained by applying the first law of thermodynamics to a fluid element of unit mass. This leads to

$$\rho \frac{Dh}{Dt} - \frac{Dp}{Dt} + \frac{\partial q_i}{\partial x_j} = 0 , \qquad (3.3)$$

where q_j is the component of the radiation heat-flux and h is the specific enthapy.

For a gas in vibrational and chemical equilibrium, the thermal and caloric equations of state for the assumed perfect gas are

$$p = \rho RT , \qquad (3.4)$$

and

$$h = \frac{\gamma}{\gamma-1}\frac{p}{\rho} , \qquad (3.5)$$

where T is the temperature of the gas.

The radiation-transport equation for a non-scattering, grey gas in local thermodynamic equilibrium is given by Eq. (2.54) as

$$\ell_j \frac{\partial I}{\partial x_j} = -\alpha\left(I - \frac{\sigma T^4}{\pi}\right) , \qquad (3.6)$$

where the unsteady term has been neglected for the reason explained in Section 2.9.

The radiation field is coupled with the flow field through the integral

$$q_i(\bar{r}, t) = \int_{\Omega} I(\bar{r}, \bar{\Omega}, t)\, \ell_i \, d\Omega \ . \qquad (3.7)$$

Equations (3.1) to (3.7) are the governing equations for a perfect, inviscid, nonconducting, nonscattering, grey-gas in local thermodynamic equilibirum. If we substitute Eq. (3.7) into Eq. (3.3), it is apparent that the governing equations are of integro-differential form. Exact analytical solutions are therefore difficult, if not impossible, to obtain.

For a thin or thick gas, equations (3.6) and (3.7) are replaced respectively by Eqs. (2.63) or (2.66). Either one of these equations together with the gas-dynamic equations (3.1) through (3.5) constitute a determinate set of purely differential equations. Thus in these asymptotic situations, the integro-differential equations reduce to purely differential form.

3.2. One-Dimensional Flow Problems and the Exponential Approximation.

For one-dimensional problems, the integral term in the differential equations can be removed by the so called "exponential approximation". With this approximation, analytical solutions valid for the full range of absorption coefficient are possible for one-dimensional problems (see Vincenti and Baldwin (1962)). Since this

scheme of approximation for the one-dimensional problem is related to the spherical-harmonic approximation that we shall discuss in great length in subsequent chapters, we shall discuss the exponential approximation briefly here.

For one-dimensional problems, the special forms of the gas-dynamic equations (3.1) through (3.5) are obvious and we need not write them down explicitly here. If a black surface with temperature T_w is located at $x_2 = 0$, and if the solution of Eq. (3.6) for the one-dimensional situation is substituted into Eq. (3.7), the resulting expression for q is given by Eq. (2.71) as

$$q(\eta) = 2\sigma \int_0^{\eta} T^4(\eta') E_2(\eta-\eta') d\eta' - 2\sigma \int_{\eta}^{\infty} T^4(\eta') E_2(\eta'-\eta) d\eta' + \sigma T_w^4 E_3(\eta) . \tag{3.8}$$

The exponential approximation consists in replacing the exponential integral function $E_2(z)$ in Eq. (3.8) by a purely exponential function of the form

$$E_2(z) = \frac{1}{3} b^2 e^{-bz} , \tag{3.9}$$

where b is a constant and cannot be uniquely determined. For example, Vincenti and Baldwin (1962) take $b = 1.562$ whereas Wick (1964) takes $b = 3/2$. As we shall see in Section 4.3, if we set $b = \sqrt{3}$, the exponential approximation is equivalent to the later spherical-harmonic approximation when that approximation is specialized to one-dimensional problems with a suitable choice of boundary conditions at the wall.

With the exponential approximation given by Eq. (3.9), the exact expression (3.8) can now be approximated by

$$q(\eta) = \frac{2b^2\sigma}{3} \int_0^\eta T^4(\eta') \, e^{-b(\eta-\eta')} \, d\eta'$$

$$- \frac{2b^2\sigma}{3} \int_\eta^\infty T^4(\eta') \, e^{-b(\eta'-\eta)} \, d\eta' + \frac{b}{3} \sigma T_w^4 \, e^{-b\eta}. \quad (3.10)$$

If Eq. (3.10) is differentiated twice and the undifferentiated equation is used to remove the integral, it can be shown that q satisfies the equation

$$\frac{d^2q}{d\eta^2} - b^2 q - 16\sigma T^3 \frac{dT}{d\eta} = 0. \quad (3.11)$$

This equation together with the one-dimensional form of Eqs. (3.1) through (3.5) are then the governing equations for one-dimensional radiating gas flow with the exponential approximation. Thus, as a result of this approximation the exact one-dimensional integro-differential equations are approximated by a set of differential equations valid for the complete range of absorption coefficient. Unfortunately, this scheme of approximation is not applicable to a multi-dimensional radiating gas.

IV APPROXIMATE RADIATION-TRANSPORT EQUATIONS AND BOUNDARY CONDITIONS

In Section 2.6, we pointed out that there is a mathematical anology between transport of neutrons and photons, and that Eq. (2.39) can also be taken as governing equation for transport of neutrons. Much effort has been devoted by nuclear physicists to the approximate solution of this integro-differential equation in connection with the design of nuclear reactors. The most useful technique for this purpose is the so-called "spherical-harmonic method". The underlying principle of this method is to replace the exact integro-differential transport equation by an infinite set of coupled equations of purely differential form.

In the flow of a radiating gas, if the scattering term is negligible compared with other terms, the radiation-transport equation itself is already in purely differential form. In spite of this, the complete set of governing equations is still of integro-differential form because the radiation heat flux in the energy equation and the radiation intensity in the transport equation are related through an integral (see Section 3.1). As we shall see, however, by applying the spherical-harmonic method to the radiation-transport equation, the first approximation of the resultant equations together with the gasdynamic equations constitute a determinate set of purely differential equations valid for the complete range of absorption coefficient.

4.1. Approximate Radiation-Transport Equations.

We shall now obtain the approximate radiation-transport equations for a grey gas in local thermodynamic equilibrium by means of the spherical-harmonic method. This method was first developed by Chandrasekhar (1944) for astrophysical radiative-transport problems with plane geometry. It was later studied in greater detail in connection with neutron-transport problems by Mark (1944), (1945), Marshak (1945), Davison (1957), and Weinberg and Wigner (1958). The procedure of the method is also similar to that used by Grad (1949) in kinetic theory.

Integrating Eq. (2.51) with respect to the whole range of frequency and assuming α, η, and κ are independent of frequency, we have

$$\frac{1}{c}\frac{\partial I}{\partial t} + \bar{\Omega}\cdot \text{grad } I$$

$$= \frac{\alpha\sigma T^4}{\pi} - (\alpha+\eta)\, I + \eta \int_{\Omega'} \kappa(\bar{r}, \bar{\Omega}' \rightarrow \bar{\Omega}, t)\, I(\bar{r}, \bar{\Omega}', t)\, d\Omega' . \quad (4.1)$$

Following the procedure used in the neutron-transport theory, we expand the radiation intensity in a series of spherical harmonics as follows:

$$I(\bar{r}, \bar{\Omega}, t) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} A_\ell^m(\bar{r}, t)\, Y_\ell^m(\bar{\Omega}) , \qquad (4.2)$$

where the $A_\ell^m(\bar{r},t)$'s are functions to be found and the $Y_\ell^m(\bar{\Omega})$'s are spherical harmonics (see Appendix A for a brief discussion of the spherical harmonics). We assume also that the scattering function κ is a function

only of the angle between the directions $\bar{\Omega}$ and $\bar{\Omega}'$. We call this angle θ_o and define $\cos\theta_o \equiv \mu_o = \bar{\Omega}\cdot\bar{\Omega}'$. Since the scattering function is a function of the single variable μ_o, a suitable series representation can be given in terms of the Legendre polynomial $P_\ell(\mu_o)$. Thus we have

$$\kappa(\bar{r}, \bar{\Omega}' \to \bar{\Omega}, t) = \kappa(\mu_o, \bar{\Omega}') = \sum_{\ell=0}^{\infty} \kappa_\ell P_\ell(\cos\theta_o) \quad . \tag{4.3}$$

We now substitute Eqs. (4.2) and (4.3) into Eq. (4.1). If the resultant equation is multiplied by $\bar{Y}_\ell^m(\bar{\Omega})$, the complex conjugate of $Y_\ell^m(\bar{\Omega})$, and integrated over the entire range of solid angle, Eq. (4.1) becomes, after application of the recurrence and orthogonality relations for the spherical harmonics (see Appendix B for details),

$$\begin{aligned}
&\frac{1}{c}\frac{\partial A_\ell^m}{\partial t} - \frac{[(\ell+m+2)\ (\ell+m+1)]^{1/2}}{2(2\ell+3)}\left[\frac{\partial A_{\ell+1}^{m+1}}{\partial x_3} + i\frac{\partial A_{\ell+1}^{m+1}}{\partial x_1}\right] \\
&\quad + \frac{[(\ell-m)\ (\ell-m-1)]^{1/2}}{2(2\ell-1)}\left[\frac{\partial A_{\ell-1}^{m+1}}{\partial x_3} + i\frac{\partial A_{\ell-1}^{m+1}}{\partial x_1}\right] \\
&\quad + \frac{[(\ell-m+2)\ (\ell-m+1)]^{1/2}}{2(2\ell+3)}\left[\frac{\partial A_{\ell+1}^{m-1}}{\partial x_3} - i\frac{\partial A_{\ell+1}^{m-1}}{\partial x_1}\right] \\
&\quad - \frac{[(\ell+m)\ (\ell+m-1)]^{1/2}}{2(2\ell-1)}\left[\frac{\partial A_{\ell-1}^{m-1}}{\partial x_3} - i\frac{\partial A_{\ell-1}^{m-1}}{\partial x_1}\right] \\
&\quad + \frac{[(\ell+m+1)\ (\ell-m+1)]^{1/2}}{(2\ell+3)}\ \frac{\partial A_{\ell+1}^m}{\partial x_2} + \frac{[(\ell+m)\ (\ell-m)]^{1/2}}{(2\ell-1)}\ \frac{\partial A_{\ell-1}^m}{\partial x_2} \\
&= \frac{\eta 4\pi\kappa_\ell A_\ell^m}{(2\ell+1)} - (\alpha+\eta)\, A_\ell^m + \frac{\alpha\sigma T^4}{\pi}\ \delta_{0m}\,\delta_{0\ell}\ ,
\end{aligned} \tag{4.4}$$

where δ_{0m} and $\delta_{0\ell}$ are the Kronecker delta. Equation (4.4) represents an infinite set of differential equations for an infinite number of unknown functions A_ℓ^m. This set is completely equivalent to the integro-differential equation (4.1).

We can obtain an N-th approximation to the foregoing set (called the P_N-approximation in neutron-transport theory) by truncating the series (4.2) after the term $\ell=N$. It is known from neutron-transport theory that an odd numbered approximation is more accurate than the succeeding even-numbered approximation and that the first approximation, the P_1-approximation (often called the diffusion approximation), is sufficiently accurate for most problems. For our purpose, therefore, we limit ourselves for the present to the first approximation. For the first approximation (N=1), Eq. (4.4) with $A_\ell^m = 0$ for $\ell \geq 2$ gives

$$\ell = 0,\ m = 0 : \frac{1}{c}\frac{\partial A_0^0}{\partial t} - \frac{\sqrt{2}}{6}\left[\frac{\partial A_1^1}{\partial x_3} + i\frac{\partial A_1^1}{\partial x_1}\right] + \frac{\sqrt{2}}{6}\left[\frac{\partial A_1^{-1}}{\partial x_3} - i\frac{\partial A_1^{-1}}{\partial x_1}\right] + \frac{1}{3}\frac{\partial A_1^0}{\partial x_2}$$

$$= 4\pi\kappa_0\eta A_0^0 - (\alpha+\eta)A_0^0 + \frac{\alpha\sigma T^4}{\pi}, \qquad (4.5a)$$

$$\ell = 1,\ m = -1 : \frac{1}{c}\frac{\partial A_1^{-1}}{\partial t} + \frac{1}{\sqrt{2}}\left[\frac{\partial A_0^0}{\partial x_3} + i\frac{\partial A_0^0}{\partial x_1}\right] = \frac{4\pi}{3}\kappa_1\eta A_1^{-1} - (\alpha+\eta)A_1^{-1}, \qquad (4.5b)$$

$$\ell = 1,\ m = 0 : \frac{1}{c}\frac{\partial A_1^0}{\partial t} + \frac{\partial A_0^0}{\partial x_2} = \frac{4\pi}{3}\kappa_1\eta\, A_1^0 - (\alpha+\eta)\, A_1^0\ , \tag{4.5c}$$

$$\ell = 1,\ m = 1 : \frac{1}{c}\frac{\partial A_1^1}{\partial t} - \frac{1}{\sqrt{2}}\left[\frac{\partial A_0^0}{\partial x_3} - i\,\frac{\partial A_0^0}{\partial x_1}\right] = \frac{4\pi}{3}\kappa_1\eta\, A_1^1 - (\alpha+\eta)\, A_1^1\ . \tag{4.5d}$$

Thus for the first approximation, if T, κ_0, and κ_1 are known, we have in Eqs. (4.5), four equations for the four unknowns A_0^0, A_1^{-1}, A_1^0, A_1^1. The quantities κ_0 and κ_1 are obtained as follows from Eq. (4.3) by utilization of the orthogonality property of the spherical harmonics together with Eq. (2.34):

$$\kappa_0 = \frac{1}{4\pi}\int_{\Omega'} \kappa(\mu_0, \bar{\Omega}')\, Y_0^0(\bar{\Omega}')\, d\Omega' = \frac{1}{4\pi}\int_{\Omega'} \kappa(\mu_0, \bar{\Omega}')\, d\Omega' = \frac{1}{4\pi}\ , \tag{4.6a}$$

$$\kappa_1 = \frac{3}{4\pi}\int_{\Omega'} Y_1^0(\bar{\Omega}')\, \kappa(\mu_0, \bar{\Omega}')\, d\Omega' = \frac{3}{4\pi}\,\frac{\int_{\Omega'} \mu_0\, \kappa(\mu_0, \bar{\Omega}')\, d\Omega'}{\int_{\Omega'} \kappa(\mu_0, \bar{\Omega}')\, d\Omega'} \equiv \frac{3}{4\pi}\,\bar{\mu}\ , \tag{4.6b}$$

where $\bar{\mu}$ is the average cosine of scattering. In the case of spherically symmetric scattering, $\bar{\mu} = 0$, consequently, $\kappa_1 = 0$.

Equations (4.5) can be rewritten in terms of the space-integrated radiation intensity and the radiation heat-flux as follows: If we substitute the series (4.2) into Eqs. (2.43a) and (2.43c) and utilize the orthogonality property of the spherical harmonics, it can be shown that

$$I_0(\bar{r}, t) = 4\pi A_0^0(\bar{r}, t) , \tag{4.7a}$$

$$q_1(\bar{r}, t) = - i \frac{2\sqrt{2\pi}}{3} \left[A_1^{-1}(\bar{r},t) + A_1^1(\bar{r}, t)\right] , \tag{4.7b}$$

$$q_2(\bar{r}, t) = \frac{4\pi}{3} A_1^0(\bar{r}, t) , \tag{4.7c}$$

$$q_3(\bar{r}, t) = \frac{2\sqrt{2\pi}}{3} \left[A_1^{-1}(\bar{r},t) - A_1^1(\bar{r}, t)\right] . \tag{4.7d}$$

In view of Eqs. (4.6a) and (4.7), Eq. (4.5a) can be rewritten as

$$\frac{1}{c}\frac{\partial I_0}{\partial t} + \frac{\partial q_j}{\partial x_j} = - \alpha(I_0 - 4\sigma T^4) \quad , \tag{4.8a}$$

and with some algebraic manipulation, Eqs. (4.5b) through (4.5d) can be rewritten as

$$\frac{1}{c}\frac{\partial q_i}{\partial t} + \frac{\partial I_0}{\partial x_i} = -3(\alpha + \eta - \bar{\mu}\eta)\, q_i, \;\; (i = 1,2,3). \tag{4.8b}$$

Note that the scattering coefficient enters only on the right-hand side of Eq. (4.8b). Thus the effect of scattering is equivalent to a modification of the coefficient on the right-hand side of Eq. (4.8b).

If the unsteady term is neglected for the reason explained in Section 2.9, the approximate radiation-transport equations (4.8) for a non-scattering, grey gas in local thermodynamic equilibrium become

$$\frac{\partial q_j}{\partial x_j} = -\alpha(I_o - 4\sigma T^4) \quad , \tag{4.9a}$$

and

$$\frac{\partial I_o}{\partial x_i} = - 3\alpha q_i, \qquad (i = 1, 2, 3) \quad . \tag{4.9b}$$

These equations can be rewritten in vector form as

$$\text{div}\ \bar{q} \equiv \nabla \cdot \bar{q} = -\alpha(I_o - 4\sigma T^4) \quad , \tag{4.10a}$$

and

$$\text{grad}\ I_o \equiv \nabla\ I_o = -3\alpha\bar{q} \quad . \tag{4.10b}$$

Equations (4.9) or Eqs. (4.10) are identical to the four approximate radiation-transport equations previously obtained by the author (1964) by means of a moment method. If I_o is eliminated from Eqs. (4.9) or Eqs. (4.10), the resulting equations (except for sign conventions and notation) are identical to the equations cited without proof by Traugott (1963).

With the aid of Eq. (4.7) and Eq. (A.4) in Appendix A, the radiation intensity in the expression (4.2) for the first approximation can be written

$$I(x_1,\ x_2,\ x_3,\ \phi,\ \theta,\ t)$$

$$= \frac{1}{4\pi}\left[I_o(x_1,x_2,x_3,t) + 3q_1 \sin\phi \sin\theta + 3q_2 \cos\theta + 3q_3 \cos\phi \sin\theta\right]. \tag{4.11a}$$

The vector form of this equation, in view of Eq. (2.18), can be written as

$$I(\bar{r},\ \bar{\Omega},\ t) = \frac{1}{4\pi}\left[I_o(\bar{r},\ t) + 3\bar{q} \cdot \bar{\Omega}\right]. \tag{4.11b}$$

With the aid of Eq. (4.10b), this equation can also be rewritten as

$$I(\bar{r},\bar{\Omega},t) = \frac{1}{4\pi}\left[I_o(\bar{r},t) - \frac{1}{\alpha}\bar{\Omega} \cdot \text{grad } I_o\right]. \tag{4.11c}$$

As we shall see in Section 4.2, either one of Eqs. (4.11) is required for use in connection with the boundary conditions.

It is of interest to note that Eqs. (4.9) reduce to the usual thin- or thick-gas approximation when the mean free path of the photons $(1/\alpha)$ is much larger or smaller than the characteristic dimension ℓ of the problem. This can be shown by converting both the dependent and independent variables in Eqs. (4.9) to dimensionless quantities of order unity. It then becomes apparent that for $\alpha\ell \ll 1$ the absorption term in Eq. (4.9a) is of higher order than the rest of the terms. Thus for $\alpha\ell \ll 1$, Eq. (4.9a) gives

$$\frac{\partial q_j}{\partial x_j} = 4\alpha\sigma T^4 \quad . \tag{4.12}$$

On the other hand, if $\alpha\ell \gg 1$, the order-of-magnitude estimate shows that the term on the left in Eq. (4.9a) is negligible compared with other terms. This leads to

$$I_o = 4\sigma T^4 \quad . \tag{4.13}$$

With the aid of this result, Eq. (4.9b) then gives

$$q_i = - \frac{16\sigma T^3}{3\alpha} \frac{\partial T}{\partial x_i} \quad . \tag{4.14}$$

Equations (4.12) and (4.14) are the well-known thin- and thick-gas (Rosseland) approximations. They thus agree with what is obtained from asymptotic expansion of the solution of the exact radiation-transport equation as discussed in Section 2.9.

For problems involving only one spatial dimension, say x_2, the corresponding P_1-approximation for the radiation-transport equation and the radiation intensity in this case are obtained by setting q_1 and q_3 equal to zero in Eqs. (4.9) and (4.11). These results can also be obtained formally by returning to the exact radiation-transport equation for one-dimensional problem and expanding the radiation intensity in terms of Legendre polynomials. Since the spherical-harmonic series with $m = 0$ reduces to a series in terms of Legendre polynomials,

we can carry out the procedure alternatively by putting $A_\ell^m = 0$ for $m \neq 0$ in Eqs. (4.2) and (4.4). This leads to the series

$$I(x,\mu,t) = \sum_{\ell=0}^{\infty} A_\ell(x,t)\, P_\ell(\mu) \quad , \tag{4.15}$$

where the upper index 0 on the right-hand side of Eq. (4.15) and the subscript of x have been dropped. The infinite set of radiation-transport equations for problems with plane geometry become

$$\frac{1}{c}\,\frac{\partial A_\ell}{\partial t} + \frac{(\ell+1)}{(2\ell+3)}\,\frac{\partial A_{\ell+1}}{\partial x} + \frac{\ell}{(2\ell-1)}\,\frac{\partial A_{\ell-1}}{\partial x}$$

$$= \frac{\eta 4\pi\kappa_\ell\, A_\ell}{(2\ell+1)} - (\alpha+\eta)\, A_\ell + \frac{\alpha\sigma T^4}{\pi}\,\delta_{0\ell} \quad . \tag{4.16}$$

For the first approximation (P_1-approximation), we set A_2, A_3, ... equal to zero in Eqs. (4.15) and (4.16). The resulting equations can then be put in terms of the space-integrated intensity and the radiation heat flux, which are given by Eqs. (4.7a) and (4.7c) as

$$I_o(x,\, t) = 4\pi\, A_o(x,\, t) \;, \tag{4.17a}$$

and

$$q(x,\, t) = \frac{4\pi}{3}\, A_1(x,\, t), \tag{4.17b}$$

where the subscript of q in Eq. (4.17b) has been dropped.

In view of Eqs. (4.6) and (4.17), the P_1-approximation of Eqs. (4.15) and (4.16) are

$$I(x, \mu, t) = \frac{1}{4\pi}[I_o(x,t) + 3\mu\, q(x,t)], \qquad (4.18)$$

and

$$\frac{1}{c}\frac{\partial I_o}{\partial t} + \frac{\partial q}{\partial x} = -\alpha(I_o - 4\sigma T^4) \quad , \qquad (4.19a)$$

$$\frac{1}{c}\frac{\partial q}{\partial t} + \frac{\partial I_o}{\partial x} = -3(\alpha+\eta-\bar{\mu}\eta)\, q \quad . \qquad (4.19b)$$

If the unsteady term and the effect of scattering are neglected, Eqs. (4.19) become

$$\frac{\partial q}{\partial x} = -\alpha(I_o - 4\sigma T^4) \quad , \qquad (4.20a)$$

$$\frac{\partial I_o}{\partial x} = -3\alpha q \quad . \qquad (4.20b)$$

Equations (4.20) are the one-dimensional case of Eqs. (4.9). Except for sign convention and notation, these equations are identical to those obtained by Traugott (1963) for the first approximation by means of a moment method. The P_3-approximation of Eq. (4.16), neglecting the effect of scattering, is essentially the second approximation obtained by Traugott (1963).

4.2. Approximate Boundary Conditions.

If a surface is present in the flow field, the surface condition and the distribution of wall temperature will effect the radiation and flow fields. Suppose a black surface with outward-drawn unit normal vector $\bar{n}$ is described by the equation $F(\bar{r}) = 0$. The radiation intensity at the wall is therefore given by

$$I^{+}(\bar{r}, \bar{\Omega}, t) = \frac{\sigma T_w^4(\bar{r},t)}{\pi} \qquad \text{for } \bar{\Omega} \cdot \bar{n} > 0 , \quad (4.21)$$

where $\bar{r}$ is on the prescribed surface and T_w is the local wall temperature. When the temperature field is known, the exact solution for I is uniquely determined by Eq. (4.1) subject to boundary condition (4.21). If the spherical-harmonic method is applied, the integro-differential equation (4.1) is now replaced by an infinite set of purely differential equations in terms of A_ℓ^m as given by Eq. (4.4). The corresponding infinite set of boundary conditions in terms of A_ℓ^m can be obtained by substituting Eq. (4.2) into Eq. (4.21).

The infinite number of conditions as given in Eq. (4.21) cannot be exactly satisfied in an approximation of finite order. Thus for the P_N- approximation, another set of approximate conditions consistent with the spherical-harmonic method will have to be found. This approximate set of conditions is not uniquely defined. The most frequently applied boundary conditions are those which were proposed by Mark (1944) and Marshak (1947). Both of these boundary conditions were originally proposed for neutron-transport problems with plane geometry. We shall now consider the application of Mark's and Marshak's boundary conditions to radiation-transport problems with plane geometry.

In the special case of a one-dimensional problem with a black wall located at $x_2 = 0$, the boundary condition (4.21) can be written

$$I^+(0, \mu, t) = \frac{\sigma T_w^4(t)}{\pi} \qquad \text{for } \mu > 0 \ , \qquad (4.22)$$

where $\mu \equiv \cos\theta$. Mark proposed that for the P_N-approximation this condition is satisfied for certain discrete values of μ that are taken to be the roots of the Legendre polynomial $P_{N+1}(\mu_i) = 0$. Mark's boundary condition in the P_N-approximation is thus written

$$I^+(0, \mu_i, t) = \frac{\sigma T_w'(t)}{\pi} \qquad \text{for } \mu_i > 0 \text{ and given by } P_{N+1}(\mu_i) = 0, \qquad (4.23)$$

where I^+ on the left-hand side is to be represented by the truncated series (4.15). We note that Eq. (4.23) tends to the exact boundary condition (4.22) as N approaches infinity.

Marshak suggested alternatively that for the P_N-approximation, the boundary condition (4.22) be approximated by

$$\int_0^1 I^+(0, \mu, t)\, P_{2i-1}(\mu)\, d\mu = \int_0^1 \frac{\sigma T_w^4}{\pi}(t)\, P_{2i-1}(\mu)\, d\mu$$

$$\left(i = 1, 2, \ldots, \tfrac{1}{2}(N+1)\right), \qquad (4.24)$$

with I^+ represented by the truncated series (4.15).

We now discuss the P_1-approximation of Mark's and Marshak's boundary conditions. For the P_1-approximation, the radiation intensity for problems with plane geometry is given by Eq. (4.18). Substituting this truncated series into the left-hand side of Eq. (4.23) and noting that the positive root of $P_2(\mu_1) = 3\mu_1^2 - 1 = 0$ is $1/\sqrt{3}$, we have

$$\frac{1}{4\pi}\left[I_o(0,\ t) + \sqrt{3}\, q(0,t)\right] = \frac{\sigma T_w^4(t)}{\pi}, \qquad (4.25)$$

which is Mark's boundary condition for the P_1-approximation.

For the P_1-approximation, the left-hand side of Eq. (4.24) represents the heat flux emitted from the wall. If we substitute Eq. (4.18) into Eq. (4.24) and evaluate the integrals, we have

$$\frac{1}{4}\left[I_o(0,\ t) + 2q(0,\ t)\right] = \sigma T_w^4(t), \qquad (4.26)$$

which is Marshak's boundary condition for the P_1-approximation. Equations (4.25) and (4.26) can be both written in the form

$$I_o(0,\ t) + mq(0,\ t) = 4\sigma T_w^4(t), \qquad (4.27)$$

where $m = \sqrt{3}$ for Mark's boundary condition and $m = 2$ for Marshak's boundary condition.

It is evident from the foregoing discussion that in the first approximation, Mark's boundary condition specifies the radiation intensity to be satisfied at a specific value of μ, i.e., at $\mu = 1/\sqrt{3}$ whereas Marshak's boundary condition specifies the heat flux emitted from the wall.

It has been found in neutron-transport theory that in a given approximation the difference in results obtained by applying Mark's or Marshak's boundary conditions is insignificantly small.

While the extension of Mark's boundary condition to multi-dimensional problems is not apparent, the generalization of Marshak's boundary condition leads to further complications and ambiguities. For a general treatement of this matter, the reader is referred to Davison (1957, p. 167). For our purpose, it is sufficient to quote without proof the P_1-approximation of the generalized Marshak's boundary condition.

To be specific, consider a planar wall located at $x_2 = 0$. The radiation intensity at the wall is then given by

$$I^+(x_1, 0, x_3, \phi, \theta, t) = \frac{\sigma T_w^4}{\pi}(x_1, x_3, t). \qquad (4.28)$$

The P_1-approximation of the Marshak-type boundary condition becomes

$$\int_{\theta=0}^{\pi/2} \int_{\phi=0}^{\phi=2\pi} I^+ \bar{Y}_1^0(\bar{\Omega})\, d\Omega = \int_{\theta=0}^{\pi/2} \int_{\phi=0}^{\phi=2\pi} \frac{\sigma T_w^4}{\pi} \bar{Y}_1^0(\bar{\Omega})\, d\Omega \,. \qquad (4.29)$$

In view of Eqs. (2.23) and (A-4a) in Appendix A, the left-hand side of Eq. (4.29) represents the heat flux emitted from the wall. Substituting Eq. (4.11a) into Eq. (4.29) and evaluating the integrals, we have

$$\frac{1}{4}\,[I_o(x_1, 0, x_3, t) + 2q_2(x_1, 0, x_3, t)] = \sigma T_w^4(x_1, x_3, t). \qquad (4.30)$$

This is the P_1-approximation of the Marshak's boundary condition for multi-dimensional problems. Comparing Eqs. (4.26) and (4.30), we note that these equations are of the same form and that Eq. (4.26)

is a special case of Eq. (4.30). Equation (4.30) provides a suitable radiation boundary condition for the planar wall in a three-dimensional flow field.

4.3. The Relation of the Exponential Approximation and the Spherical-Harmonic Approximation.

It is shown in this section that the exponential approximation discussed in Section 3.2 is equivalent to the P_1-approximation when that approximation is specialized to one-dimensional problems and used with Mark's boundary condition. We are concerned here specifically with the one-dimensional problem posed in the previous section with boundary condition given by Eq. (4.22). To begin, we note that if I_o is eliminated from Eqs. (4.20) the resulting equation is identical to Eq. (3.11) with $b = \sqrt{3}$. This, however, is not sufficient to prove that the two schemes of approximation are equivalent, since the information on boundary conditions is lost in the exponential approximation when the integral equation is converted into differential equation.

In order to prove the equivalence, we first eliminate q from Eqs. (4.20) to obtain

$$\frac{d^2 I_o}{d\eta^2} - 3I_o + 12\sigma T^4 = 0 \ . \qquad (4.31)$$

The general solution of this equation can be shown to be

$$I_o(\eta) = 2\sqrt{3}\sigma \int_0^\eta T^4(\eta') \, e^{\sqrt{3}\,(\eta'-\eta)} \, d\eta' \qquad (4.32)$$

$$+ 2\sqrt{3}\sigma \int_\eta^\infty T^4(\eta') \, e^{\sqrt{3}\,(\eta-\eta')} \, d\eta' + C_1 \, e^{-\sqrt{3}\eta} + C_2 \, e^{\sqrt{3}\eta} ,$$

where $C_1(t)$ and $C_2(t)$ are to be determined from the boundary conditions. If I_o is finite at infinity, $C_2(t)$ must vanish. Substitution of Eq. (4.32) with $C_2(t) = 0$ into Eq. (4.20b) yields

$$q(\eta) = 2\sigma \int_0^\eta T^4(\eta') e^{\sqrt{3}(\eta'-\eta)} d\eta'$$

$$- 2\sigma \int_\eta^\infty T^4(\eta') e^{\sqrt{3}(\eta-\eta')} d\eta' + \frac{C_1}{\sqrt{3}} e^{-\sqrt{3}\eta} . \qquad (4.33)$$

Imposing Mark's boundary condition (4.25) and utilizing Eqs. (4.32) and (4.33), we find

$$C_1(t) = 2\sigma T_w^4 . \qquad (4.34)$$

Equations (4.32) and (4.33) together with expression (4.34) provide a solution for I_o and q as obtained from the P_1-approximation specilized to the one-dimensional problem and utilizing Mark's boundary condition. If we compare these results with the exact solution given in Eqs. (2.70) and (2.71), it is evident that the P_1-approximation with Mark's boundary condition is equivalent to putting $b = \sqrt{3}$ and hence

$$E_2(z) = e^{-\sqrt{3}z} , \qquad (4.35)$$

in the exact solution for the one-dimensional problem. It should be emphazied that our conclusion is rather general and is independent of the temperature distribution in the flow field.

V. LINEARIZED THEORY

It was shown in Section 4.1 that to a first approximation the exact multi-dimensional transport equation can be replaced by four differential equations in terms of the unknowns I_o and q_i. The approximate radiation-transport equations given by Eqs. (4.9) together with the gasdynamic equations (3.1) through (3.5) constitute a determinate set of purely differential equations valid for the complete range of absorption coefficient. Although the governing equations are in differential form, they are nonlinear, and analytical solution is therefore still difficult. In this chapter, we shall examine the corresponding linearized equations.

5.1. Acoustic Equations.

We are concerned here with the problem of weak disturbances propagating into a uniform medium that is in radiation equilibrium. If the coordinate system is fixed in the undisturbed fluid, the dependent variables are given by $u_i = u_i'$, $p = p_\infty + p'$, $\rho = \rho_\infty + \rho'$, $\alpha = \alpha_\infty + \alpha'$, $I_o = I_{o_\infty} + I_o'$, $q_i = q_i'$ etc., where the disturbance quantities are denoted by primes and the subscript ∞ denotes the equilibrium reference condition. Substituting these variables into the governing equations (3.1) through (3.5) and (4.9), and proceeding with the linearization in the usual fashion, we obtain

$$\frac{\partial \rho'}{\partial t} + \rho_\infty \frac{\partial u_j'}{\partial x_j} = 0 \ , \qquad (5.1)$$

$$\rho_\infty \frac{\partial u_i'}{\partial t} + \frac{\partial p'}{\partial x_i} = 0 \qquad (i = 1,\ 2,\ 3), \qquad (5.2)$$

$$\rho_\infty \frac{\partial h'}{\partial t} - \frac{\partial p'}{\partial t} + \frac{\partial q_j'}{\partial x_j} = 0 \ , \qquad (5.3)$$

$$T' = \frac{1}{R}\left[\frac{p'}{\rho_\infty} - \frac{p_\infty}{\rho_\infty^2}\rho' \right] \ , \qquad (5.4)$$

$$h' = \frac{\gamma}{\gamma - 1}\left[\frac{p'}{\rho_\infty} - \frac{p_\infty}{\rho_\infty^2}\rho' \right] , \qquad (5.5)$$

$$\frac{\partial q_j'}{\partial x_j} = - \alpha_\infty (I_o' - 16\sigma T_\infty^3 \, T') \ , \qquad (5.6a)$$

$$\frac{\partial I_o'}{\partial x_i} = - 3\alpha_\infty \, q_i' \qquad (i = 1,\ 2,\ 3) \ . \qquad (5.6b)$$

We now introduce a perturbation velocity potential ϕ such that

$$u_i' = \frac{\partial \phi}{\partial x_i} \qquad \text{and} \qquad p' = -\rho_\infty \frac{\partial \phi}{\partial t} \ . \qquad (5.7)$$

These expressions satisfy the momentum equations (5.2). The remaining equations (5.1) through (5.6) can then be reduced to a single equation in terms of ϕ. To this end, we first differentiate Eq. (5.4) with respect to t and make use of Eqs. (5.1) and (5.7) to obtain

$$\frac{\partial T'}{\partial t} = - \frac{1}{R} \left[\frac{\partial^2 \phi}{\partial t^2} - a_T^2 \frac{\partial^2 \phi}{\partial x_j \, \partial x_j} \right] , \qquad (5.8)$$

where $a_T \equiv \sqrt{RT_\infty}$ is the isothermal speed of sound. Eliminating h' from Eqs. (5.3), (5.5) and making use of Eqs. (5.1), (5.7) and (5.6a), we have

$$\frac{\rho_\infty}{\gamma-1} \left[\frac{\partial \phi}{\partial t} - a_S^2 \frac{\partial^2 \phi}{\partial x_j \, \partial x_j} \right] = - \alpha_\infty (I_o' - 16 \sigma T_\infty^3 T'), \qquad (5.9)$$

where $a_S \equiv \sqrt{\gamma RT_\infty}$ is the isentropic speed of sound. If q_j' is eliminated from Eqs. (5.6), we obtain

$$\frac{\partial}{\partial x_j} \left(\frac{\partial I_o'}{\partial x_j} \right) - 3\alpha_\infty^2 \, I_o' + 48\alpha_\infty \, \sigma T_\infty^3 T' = 0 \ . \qquad (5.10)$$

Finally, Eqs. (5.8), (5.9) and (5.10) are combined to give the following single equation in terms of ϕ:

$$\frac{\partial^3 W_S}{\partial t \, \partial x_j \, \partial x_j} + \frac{16 \gamma a_S \alpha_\infty}{Bo} \frac{\partial^2 W_T}{\partial x_j \, \partial x_j} - 3\alpha_\infty^2 \frac{\partial W_S}{\partial t} = 0 \ , \qquad (5.11)$$

where $W_S \equiv \frac{\partial^2 \phi}{\partial t^2} - a_S^2 \frac{\partial^2 \phi}{\partial x_j \, \partial x_j}$, $W_T \equiv \frac{\partial^2 \phi}{\partial t^2} - a_T^2 \frac{\partial^2 \phi}{\partial x_j \, \partial x_j}$, and

$Bo \equiv \dfrac{\gamma R \rho_\infty \, a_S}{(\gamma-1) \, \sigma T_\infty^3}$. The dimensionless quantity Bo is called the

Boltzmann number.*

Equation (5.11) is a fifth-order linear partial differential equation in ϕ. With appropriate boundary conditions, ϕ can be determined. The disturbed velocity and pressure fields can then be found from Eqs. (5.7). The disturbed radiation field in terms of ϕ can be found as follows: Eliminating I_o' from (5.6b) and (5.9) and differentiating the resulting equation with respect to t, we have, in view of Eq. (5.8),

$$\frac{\partial q_i'}{\partial t} = \frac{\rho_\infty}{3\alpha_\infty^2(\gamma-1)} \frac{\partial^2 W_S}{\partial t\, \partial x_i} + \frac{16\sigma T_\infty^3}{3\alpha_\infty R} \frac{\partial W_T}{\partial x_i} \, . \tag{5.12}$$

Differentiating Eq. (5.6a) with respect to t, we have, in view of Eqs. (5.8) and (5.12),

$$\frac{\partial I_o'}{\partial t} = 16\sigma T_\infty^3 \frac{\partial T'}{\partial t} - \frac{\rho_\infty}{3\alpha_\infty^3(\gamma-1)} \frac{\partial^3 W_S}{\partial t\, \partial x_j\, \partial x_j} - \frac{16\sigma T_\infty^3}{3\alpha_\infty^2 R} \frac{\partial^2 W_T}{\partial x_j\, \partial x_j} \, . \tag{5.13a}$$

With the aid of Eq. (5.11), this equation can also be written

$$\frac{\partial I_o'}{\partial t} = 16\sigma T_\infty^3 \frac{\partial T'}{\partial t} - \frac{\rho_\infty}{\alpha_\infty(\gamma-1)} \frac{\partial W_S}{\partial t} = - \frac{16\sigma T_\infty^3}{R} W_T - \frac{\rho_\infty}{\alpha_\infty(\gamma-1)} \frac{\partial W_S}{\partial t} \, . \tag{5.13b}$$

As we shall see, Eqs. (5.12) and (5.13) are needed in formulating the boundary condition at the wall.

---

*By using the exponential approximation applicable in the one-dimensional case, Lick (1964) obtained an equation that corresponds to Eq. (5.11) specialized to unsteady one-dimensional flow.

Equation (5.11) plays the same role here as does the wave equation in classical acoustic theory. Since Eq. (5.11) is of higher order than the classical equation, additional boundary conditions are necessary. If a radiating wall exists in the flow field, the surface condition as well as the distribution of wall temperature will affect the flow field.

To simplify the analysis, we assume that the wall is black. Since viscosity and heat conductivity are neglected, the temperature of the wall is not necessarily equal to the temperature of the gas immediately adjacent to the wall. Consider a planar black wall with local temperature T_w located at $x_2 = 0$. The relation governing the temperature jump at the wall can be obtained by utilizing Eq. (4.30) as follows: Linearization of Eq. (4.30) leads first to the form

$$16\sigma T_\infty^3 \, T_w'(x_1, x_3, t) = I_o'(x_1, 0, x_3, t) + 2q_2'(x_1, 0, x_3, t). \quad (5.14)$$

Differentiating this relation with respect to t and substituting Eqs. (5.12) and (5.13b) into the resulting equation, we obtain finally

$$\left[\frac{\partial T_w'}{\partial t}(x_1, x_3, t) - \frac{\partial T'}{\partial t}(x_1, 0, x_3, t) \right]$$

$$= \frac{Bo}{16\alpha_\infty \gamma Ra_S} \frac{\partial}{\partial t} \left[\frac{2}{3\alpha_\infty} \frac{\partial W_S}{\partial x_2} - W_S \right]_{(x_1,0,x_3,t)}$$

$$+ \frac{2}{3\alpha_\infty R} \left(\frac{\partial W_T}{\partial x_2} \right)_{(x_1,0,x_3,t)} \quad . \quad (5.15a)$$

This equation shows clearly the temperature jump at the wall. For an opaque gas ($\alpha_\infty \to \infty$), Eq. (5.15a) becomes

$$\frac{\partial T'_w}{\partial t}(x_1,x_3,t) = \frac{\partial T}{\partial t}(x_1, 0, x_3, t) \quad . \tag{5.15b}$$

Thus for an opaque gas no temperature jump exists at the wall.

In view of Eq. (5.8), Eq. (5.15a) can also be written as

$$\frac{\partial T'_w}{\partial t}(x_1, x_3, t)$$

$$= \frac{Bo}{16\alpha_\infty \gamma Ra_S} \frac{\partial}{\partial t}\left[\frac{2}{3\alpha_\infty} \frac{\partial W_S}{\partial x_2} - W_S\right]_{(x_1,0,x_3,t)} + \frac{1}{R}\left[\frac{2}{3\alpha_\infty}\frac{\partial W_T}{\partial x_2} - W_T\right]_{(x_1,0,x_3,t)} \quad . \tag{5.15c}$$

This provides the radiative boundary condition at the wall.

5.2. Galilean Transformation; Steady Flow.

For problems involving a body moving with constant speed and attitude in a medium at rest, it is convenient to introduce a coordinate system fixed in the body. In such a coordinate system, the problem transforms to one of steady flow about a stationary body. If the body is slender, a single equation in terms of ϕ can be obtained by returning to the governing equations (3.1) through (3.5) and (4.9) and proceeding with the linearization accordingly. The desired results can also be obtained from the preceding section , however, by a Galilean transformation. If the body is moving with constant speed U_∞ in the

positive x_1-direction, Eq. (5.11) after such a transformation becomes

$$\frac{\partial^3 L_S}{\partial x_1\, \partial x_j\, \partial x_j} + \frac{16\alpha_\infty}{Bo}\, \frac{\partial^2 L_T}{\partial x_j\, \partial x_j} - 3\alpha_\infty^2 \frac{\partial L_S}{\partial x_1} = 0 \,, \tag{5.16}$$

where $L_S \equiv (1-M_{S_\infty}^2)\dfrac{\partial^2 \phi}{\partial x_1^2} + \dfrac{\partial^2 \phi}{\partial x_2^2} + \dfrac{\partial^2 \phi}{\partial x_3^2}$, $L_T \equiv (1-M_{T_\infty}^2)\dfrac{\partial^2 \phi}{\partial x_1^2} + \dfrac{\partial^2 \phi}{\partial x_2^2} + \dfrac{\partial^2 \phi}{\partial x_3^2}$,

and $Bo \equiv \dfrac{\gamma R \rho_\infty U_\infty}{(\gamma-1)\sigma T_\infty^3}$. Here $M_{S_\infty} \equiv U_\infty/a_{S_\infty}$ is the isentropic Mach number,

and $M_{T_\infty} \equiv U_\infty/a_{T_\infty} = \gamma M_{S_\infty}$ is the isothermal Mach number. With the transformation, Eqs. (5.7), (5.8), (5.12) and (5.13b) become

$$u_1' = \frac{\partial \phi}{\partial x_1} \,, \tag{5.17a}$$

$$p' = -\rho_\infty U_\infty \frac{\partial \phi}{\partial x_1} \,, \tag{5.17b}$$

$$\frac{\partial T'}{\partial x_1} = \frac{U_\infty}{\gamma R M_{S_\infty}^2} L_T \,, \tag{5.17c}$$

$$\frac{\partial q_i'}{\partial x_i} = -\frac{\rho_\infty U_\infty^2}{3\alpha_\infty^2(\gamma-1) M_{S_\infty}^2}\, \frac{\partial^2 L_S}{\partial x_i\, \partial x_i} - \frac{16\sigma T_\infty^3 U_\infty}{3\alpha_\infty \gamma R M_{S_\infty}^2}\, \frac{\partial L_T}{\partial x_i} \,, \tag{5.17d}$$

$$\frac{\partial I_0'}{\partial x_1} = 16\sigma T_\infty^3 \frac{\partial T'}{\partial x_1} + \frac{\rho_\infty U_\infty}{\alpha_\infty(\gamma-1) M_{S_\infty}} \frac{\partial L_S}{\partial x_1}$$

$$= \frac{16\sigma T_\infty^3 U_\infty}{\gamma R M_{S_\infty}^2} L_T + \frac{\rho_\infty U_\infty^2}{\alpha_\infty(\gamma-1) M_{S_\infty}^2} \frac{\partial L_S}{\partial x_1} \quad . \tag{5.17e}$$

After the Galilean transformation, Eqs. (5.15a) and (5.15c) become

$$\left[\frac{\partial T_w'}{\partial x_1}(x_1, x_3) - \frac{\partial T'}{\partial x_1}(x_1, 0, x_3)\right]$$

$$= \frac{Bo\, U_\infty}{16\alpha_\infty \gamma R M_{S_\infty}^2} \frac{\partial}{\partial x_1}\left[L_S - \frac{2}{3\alpha_\infty}\frac{\partial L_S}{\partial x_2}\right]_{(x_1,0,x_3)} - \frac{2U_\infty}{3\alpha_\infty \gamma R M_{S_\infty}^2} \frac{\partial L_T}{\partial x_2}(x_1,0,x_3) \ , \tag{5.18a}$$

and

$$\frac{\partial T_w'}{\partial x_1}(x_1, x_3) = \frac{Bo\, U_\infty}{16\alpha_\infty \gamma R M_{S_\infty}^2} \frac{\partial}{\partial x_1}\left[L_S - \frac{2}{3\alpha_\infty}\frac{\partial L_S}{\partial x_2}\right]_{(x_1,0,x_3)}$$

$$+ \frac{U_\infty}{\gamma R M_{S_\infty}^2}\left[L_T - \frac{2}{3\alpha_\infty}\frac{\partial L_T}{\partial x_2}\right]_{(x_1,0,x_3)} \quad . \tag{5.18b}$$

Equation (5.16) plays the same role in radiating gas flow as does the Prandtl-Glauert equation in classical compressible flow theory. As we recall, for supersonic flow in classical theory, the structure of the Prandtl-Glauert equation in two or three space corrdinates is similar to

that of the acoustic equation in one less space coordinate. The streamwise coordinate in the Prandtl-Glauert equation is then "time-like." From the works of Vincenti (1959), Moore and Gibson (1960), and Chu (1957), such analogy also exists in flow with chemical and vibrational non-equilibrium. The analogy does not hold, however, in the flow of a radiating gas. This can be seen by writing Eq. (5.11) explicitly for the two space coordinates x_2 and x_3 and comparing the result with Eq. (5.16) for the three space coordinates x_1, x_2 and x_3. This leads to a comparison of

$$\frac{\partial}{\partial t}\left[\frac{\partial^2}{\partial x_2^2}+\frac{\partial^2}{\partial x_3^2}\right]W_S+\frac{16\gamma a_S \alpha_\infty}{Bo}\left[\frac{\partial^2}{\partial x_2^2}+\frac{\partial^2}{\partial x_3^2}\right]W_T-3\alpha_\infty^2\frac{\partial W_S}{\partial t}=0\ , \tag{5.19a}$$

and

$$\frac{\partial}{\partial x_1}\left[\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}+\frac{\partial^2}{\partial x_3^2}\right]L_S+\frac{16\alpha_\infty}{Bo}\left[\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}+\frac{\partial^2}{\partial x_3^2}\right]L_S-3\alpha_\infty^2\frac{\partial L_S}{\partial x_1}=0\ . \tag{5.19b}$$

The structure of these equations is not similar owing to the additional terms with $\partial^2/\partial x_1^2$ in Eq. (5.19b). The reason that the analogy does not exist in the radiating gas flow is that a signal in the radiation field can propagate with the speed of light. Consequently, disturbances will propagate in the negative direction in steady flow even against a supersonic stream. On the other hand, in unsteady flow no signal can propagate in the negative direction in time.

VI. ILLUSTRATIVE EXAMPLES

To illustrate the application of the present formulation of radiating gas flow, we shall consider several simple examples. In particular, we will reconsider the problem of one-dimensional wave propagation, which has been solved previously on the basis of the exponential approximation. To provide a simple example of a multi-dimensional radiating gas flow with no restriction on absorption coefficient, we shall also consider the steady flow over a wavy wall.

6.1. Propagation of Linearized One-Dimensional Waves.

In this section, we shall consider small disturbances generated by a planar wall perpendicular to the x-axis and translating normal to itself.

Specializing Eq. (5.11) to the present problem, we have

$$\frac{\partial^3 W_S}{\partial t\, \partial x^2} + \frac{16\gamma a_S\, \alpha_\infty}{Bo} \frac{\partial^2 W_T}{\partial x^2} - 3\alpha_\infty^2 \frac{\partial W_S}{\partial t} = 0 \quad . \tag{6.1}$$

The boundary conditions imposed by the motion and temperature variation of the wall are

$$u'(0, t) = \frac{\partial \phi}{\partial x}(0, t) = \text{given function of } t, \tag{6.2a}$$

$$T_w'(t) = \text{given function of } t. \tag{6.2b}$$

The boundary condition at infinity is

$$\phi(\infty, t) = \text{finite quantity.} \tag{6.3}$$

The boundary condition (6.2b) can be expressed in terms of ϕ as follows: Since the approximate radiation boundary condition (4.26) for one-dimensional problems is a special case of Eq. (4.30), we can specialize Eqs. (5.15a) and (5.15c) for one-dimensional problems to obtain

$$\left[\frac{dT_w'(t)}{dt} - \frac{\partial T'}{\partial t}(0, t)\right]$$

$$= \frac{Bo}{16\alpha_\infty \gamma Ra_S} \frac{\partial}{\partial t}\left[\frac{m}{3\alpha_\infty}\frac{\partial W_S}{\partial x} - W_S\right]_{(0,t)} + \left[\frac{m}{3\alpha_\infty R}\frac{\partial W_T}{\partial x}\right]_{(0,t)}, \tag{6.4a}$$

and

$$\frac{dT_w'}{dt}(t) = \frac{Bo}{16\alpha_\infty \gamma Ra_S} \frac{\partial}{\partial t}\left[\frac{m}{3\alpha_\infty}\frac{\partial W_S}{\partial x} - W_S\right]_{(0,t)} + \frac{1}{R}\left[\frac{m}{3\alpha_\infty}\frac{\partial W_T}{\partial x} - W_T\right]_{(0,t)}, \tag{6.4b}$$

where m is introduced to allow for the application of Mark's boundary condition applicable to one-dimensional situations as discussed in Section 4.2. We shall now specialize the boundary conditions (6.2) to the specific problems of harmonic and impulsive motion of a planar wall, and proceed to obtain solutions for these problems.

Harmonic Motion of a Planar Wall. The problem of the propagation of disturbances generated by harmonic oscillation of a planar wall was considered by Vincenti and Baldwin (1962) with the exponential approximation. We shall now reconsider the same problem on the basis of the present differential approximation. In order to compare our results with those of Vincenti and Baldwin, it is convenient at this point to change to their variables. For this purpose, we let

$$\xi \equiv \frac{\omega x}{a_S}, \quad \beta \equiv \frac{b\alpha_\infty a_S}{\omega}, \quad K \equiv \frac{16\ b(\gamma-1)\sigma\ T_\infty^3}{3R\rho_\infty\ a_S} = \frac{16\ b\gamma}{3Bo}, \tag{6.5}$$

where ω is the radian frequency of oscillation and b is the constant used in the exponential approximation. Thus $b = 1.562$ in Vincenti and Baldwin's work, while $b = \sqrt{3}$ for the P_1-approximation as shown in Section 4.3.

The boundary conditions (6.2) for harmonic motion can be written

$$u'(0,t) = \frac{\partial\phi}{\partial x}(0,t) = \frac{RT_\infty}{a_S} \operatorname{Re}[A\ e^{i\omega t}], \tag{6.6a}$$

$$\frac{dT_w'}{dt}(t) = \omega\ T_\infty \operatorname{Re}[B\ e^{i\omega t}], \tag{6.6b}$$

where A and B are dimensionless complex constants assumed to be given.

We assume the solution in the form

$$\phi(\xi,t) = \frac{RT_\infty}{\omega} \operatorname{Re}[H(\xi)\, e^{i\omega t}] \ , \qquad (6.7)$$

where $H(\xi)$ is a function to be determined. Since Eq. (6.7) is periodic in t, substitution of this equation into Eq. (6.1) leads to a fourth-order, constant-coefficient, linear ordinary differential equation for $H(\xi)$. The solution of this equation is of the form

$$H(\xi) = \sum_{j=1}^{4} C_j \, e^{c_j \xi} \quad . \qquad (6.8)$$

Substituting Eqs. (6.7) and (6.8) into Eq. (6.1), we obtain

$$(c_j^2 - \beta^2)\,(1+c_j^2) - iK\beta\gamma^{-1}(\gamma+c_j^2)\, c_j^2 = 0 \quad . \qquad (6.9)$$

Since Eq. (6.9) is quadratic in c_j^2, half of the roots will have a positive real part. In view of the boundary condition (6.3), these roots are inadmissible. If c_3 and c_4 are the two roots with positive real part, C_3 and C_4 in Eq. (6.8) must be taken to be zero. With these considerations and imposing boundary condition (6.6a), we find

$$\sum_{j=1}^{2} c_j \, C_j = A \quad . \qquad (6.10)$$

Applying boundary condition (6.4b) and utilizing Eqs. (6.6b) and (6.9), we obtain

$$\sum_{j=1}^{2} \gamma^{-1}(\gamma + c_j^2)\, \beta \left(\frac{c_j m}{\sqrt{3}} - \beta\right) (c_j^2 - \beta^2)^{-1}\, C_j = B. \qquad (6.11a)$$

If Mark's boundary condition is used, Eq. (6.11a) with $m = \sqrt{3}$ becomes

$$\sum_{j=1}^{2} \gamma^{-1}(\gamma + c_j^2)\, \beta(\beta + c_j)^{-1}\, C_j = B\ . \qquad (6.11b)$$

Equations (6.9), (6.10), and (6.11b) are sufficient to determine all the constants in the solution (6.7). As one would anticipate from the analysis in Section 4.3, these equations are identical to Eqs. (60), (61), and (62) in Vincenti and Baldwin's paper (1962). The solution can therefore proceed as in that paper.

Impulsive motion of a planar wall. The solution for impulsive motion of a planar wall has been obtained previously by Baldwin (1962) and by Lick (1964), again using the exponential approximation. As a second example, we shall reconsider here this problem on the basis of the differential approximation. The results will be compared with those of Lick. It should be noted that, although Lick solved the same differential equation (6.1), his boundary condition corresponding to Eq. (6.4b) is in integral form.

Following Lick, we specialize boundary conditions (6.2) to

$$t < 0, \qquad u'(0,t) = \frac{\partial \phi}{\partial x}(0,t) = 0\ , \qquad (6.12a)$$

$$T_w'(t) = 0, \qquad (6.12b)$$

$$t > 0 \,, \qquad u'(0,t) = \frac{\partial \phi}{\partial x}(0,t) = \tilde{\beta} \,, \tag{6.12c}$$

$$T_w'(t) = \frac{\delta}{R} \,. \tag{6.12d}$$

The initial conditions are

$$\phi(x,0) = \frac{\partial \phi}{\partial t}(x,0) = \frac{\partial^2 \phi}{\partial t^2}(x,0) = 0 \,. \tag{6.13}$$

Let the Laplace transform of ϕ be

$$\bar{\phi}(x,\, p) = \int_0^\infty e^{-pt}\, \phi(x,t)\, dt \,. \tag{6.14}$$

Applying the Laplace transformation to Eq. (6.1) and imposing initial conditions (6.13), we obtain a fourth-order, constant-coefficient, linear ordinary differential equation for $\bar{\phi}$ The solution of this equation is of the form

$$\bar{\phi}(x,\, p) = \sum_{j=1}^{4} A_j\, e^{\gamma_j x} \,. \tag{6.15}$$

Substituting Eq. (6.15) into the ordinary differential equation, we obtain

$$p(p^2 - a_S^2\, \gamma_j^2)\,(\gamma_j^2 - 3\alpha_\infty^2) + \frac{16\, \gamma\, a_S \alpha_\infty\, \gamma_j^2}{Bo}\,(p^2 - a_T^2\, \gamma_j^2) = 0. \tag{6.16}$$

Equation (6.16) is quadratic in γ_j^2. Using the same argument as in the preceding section, we retain only the roots with negative real part. Let these two roots be denoted by γ_1 and γ_2. Solving the algebraic equation (6.16), we find

$$\gamma_{1,2} = -\left[\frac{\tilde{\delta}_2}{2\tilde{\delta}_1} \pm \frac{1}{2\tilde{\delta}_1}(\tilde{\delta}_2^2 - 4b_1^2\, p^3\, \tilde{\delta}_1)^{1/2}\right]^{1/2} \tag{6.17a}$$

where

$$\tilde{\delta}_1 \equiv a_S^2(p + a_1^2/\gamma), \quad a_1^2 \equiv \frac{16\, b^2}{3}(\gamma-1)\,\alpha_\infty \sigma T_\infty^3 = \frac{16b^3}{3Bo}\, a_S \gamma \alpha_\infty\,,$$

$$\tag{6.17b}$$

$$\tilde{\delta}_2 \equiv p(p^2 + a_1^2\, p + b_1^2\, a_S^2), \quad \text{and} \quad b_1 \equiv b\, \alpha_\infty\,.$$

In the expressions (6.17b), $b = 3/2$ was used by Lick in his exponential approximation whereas $b = \sqrt{3}$ is for the present differential approximation.

Imposing boundary condition (6.12) on (6.15), we obtain

$$\sum_{j=1}^{2} A_j\, \gamma_j = \tilde{\beta}/p\,. \tag{6.18}$$

Applying the Laplace transform to boundary conditions (6.5b) and (6.13c), and imposing these conditions on Eq. (6.15), we also obtain

$$\delta = \sum_{j=1}^{2} A_j(p^2 - a_T^2\, \gamma_j^2)\left[\frac{\gamma_j(\gamma_j - m\alpha_\infty)}{(\gamma_j^2 - 3\alpha_\infty^2)} - 1\right]\,. \tag{6.19a}$$

If Mark's boundary condition is used, Eq. (6.19a) with $m = \sqrt{3}$ becomes

$$\delta = -\sqrt{3}\,\alpha_\infty \sum_{j=1}^{2} \frac{A_j(p^2 - a_T^2\,\gamma_j^2)}{\gamma_j + \sqrt{3}\,\alpha_\infty} \quad . \tag{6.19b}$$

It follows from Eqs. (6.18) and (6.19b) that

$$A_2 = \frac{\tilde{\beta}\,C_1 - \delta\gamma_1 p}{p(\gamma_2\,C_1 - \gamma_1\,C_2)} , \tag{6.20}$$

$$A_1 = \frac{\tilde{\beta}}{\gamma_1 p} - \frac{A_2\,\gamma_2}{\gamma_1} \quad , \tag{6.21}$$

where

$$\begin{aligned} C_1 &\equiv b_1(p^2 - a_T^2\,\gamma_1^2)/(\gamma_1 + b_1), \\ C_2 &\equiv b_1(p^2 - a_T^2\,\gamma_2^2)/(\gamma_2 + b_1). \end{aligned} \tag{6.22}$$

Equations (6.15) and (6.17) with expressions (6.20) through (6.22) are equivalent to Eq. (3.2) with expressions (3.4) through (3.7) of Lick's paper (1964). Thus Lick's integral boundary condition is identical to the differential boundary condition (6.4) with $m = \sqrt{3}$ if b in his exponential approximation is suitably chosen. Again, this conclusion is to be anticipated from the analysis in Section 4.3.

6.2. Stead Flow Over a Wavy Wall.

To provide a simple analytical solution for multi-dimensional radiating flow, we now consider the steady two-dimensional flow over a wavy wall. So far as the author is aware, this is the first solution obtained for a multi-dimensional problem with no restriction on the magnitude of the absorption coefficient or temperature range.

We assume that the flow field is described by a perturbation on a uniform parallel flow with velocity U_∞ in the x_1-direction. Suppose that the wavy wall is designated by

$$X_2 = \epsilon \sin 2\pi \frac{x_1}{\ell} , \tag{6.23}$$

where ϵ is the amplitude and ℓ the wave length of the wall. To simplify the analysis, the wall is assumed to be black and to be maintained at a temperature that is a small departure from that of the gas at infinity. The governing equation for this case is given by Eq. (5.16) specialized to two-dimensional flow as follows:

$$\frac{\partial}{\partial x_1}\left[\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\right] L_S + \frac{16\alpha_\infty}{Bo}\left[\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\right] L_T - 3\alpha_\infty^2 \frac{\partial L_S}{\partial x_1} = 0, \tag{6.24}$$

where

$$L_S \equiv (1 - M_{S_\infty}^2) \frac{\partial^2 \phi}{\partial x_1^2} + \frac{\partial^2 \phi}{\partial x_2^2} \quad \text{and} \quad L_T \equiv (1 - M_{T_\infty}^2) \frac{\partial^2 \phi}{\partial x_1^2} + \frac{\partial^2 \phi}{\partial x_2^2} .$$

Within the framework of the linearized theory, the boundary conditions at the wall are

$$u_2'(x_1, 0) = \frac{\partial\phi}{\partial x_2}(x_1,0) = U_\infty \frac{dX_2}{dx_1} = 2\pi U_\infty \frac{\epsilon}{\ell} \cos 2\pi \frac{x_1}{\ell} , \qquad (6.25)$$

$$\frac{dT_w'}{dx_1}(x_1) = \frac{2\pi T_\infty}{\ell} D \cos 2\pi \frac{x_1}{\ell} , \qquad (6.26)$$

where T_w' is the perturbation wall temperature and D specifies the amplitude of the temperature variation on the wall. The fact D in general be complex allows for a possible phase shift between the wall and the variation of wall temperature. The factor $2\pi T_\infty/\ell$ in Eq. (6.26) is introduced to make D dimensionless. At infinity, we require that the disturbances are finite. Thus we have

$$\phi(x_1, \infty) = \text{finite quantity} . \qquad (6.27)$$

We now obtain the boundary condition (6.26) in terms of ϕ. Specializing Eq. (5.18b) to the two-dimensional case, we obtain the perturbation wall temperature in terms of ϕ as

$$\frac{dT_w'}{dx_1}(x_1) = \frac{\mathrm{Bo}\, U_\infty}{16\, \alpha_\infty \gamma R M_{S_\infty}^2} \frac{\partial}{\partial x_1}\left[L_S - \frac{2}{3\alpha_\infty}\frac{\partial L_S}{\partial x_2}\right]_{(x_1,0)}$$

$$+ \frac{U_\infty}{\gamma R M_{S_\infty}^2}\left[L_T - \frac{2}{3\alpha_\infty}\frac{\partial L_T}{\partial x_2}\right]_{(x_1,0)} . \qquad (6.28)$$

Imposing the boundary condition (6.26) in Eq. (6.28), we have

$$\frac{2\pi T_\infty D}{\ell} \cos 2\pi \frac{x_1}{\ell} = \frac{\text{Bo}\, U_\infty}{16\, \alpha_\infty \gamma R M_{S_\infty}^2} \frac{\partial}{\partial x_1} \left[L_S - \frac{2}{3\alpha_\infty} \frac{\partial L_S}{\partial x_2} \right]_{(x_1,0)}$$

$$+ \frac{U_\infty}{\gamma R M_{S_\infty}^2} \left[L_T - \frac{2}{3\alpha_\infty} \frac{\partial L_T}{\partial x_2} \right]_{(x_1,0)} . \qquad (6.29)$$

The periodic nature of the boundary conditions suggests that the solution is of the form

$$\phi = \frac{U_\infty \ell}{2\pi} \text{Re}[H(y)\, e^{ix}] , \qquad (6.30)$$

where the factor $U_\infty \ell/2\pi$ is introduced to make H dimensionless. The dimensionless independent variables x and y are related to the origninal variables by

$$x = \frac{2\pi x_1}{\ell} \qquad \text{and} \qquad y = \frac{2\pi x_2}{\ell}. \qquad (6.31)$$

Substitution of Eq. (6.30) into Eq. (6.24) leads to a fourth-order, constant-coefficient ordinary differential equation in H(y). The solution is of the form

$$H(y) = \sum_{j=1}^{4} A_j\, e^{c_j y} , \qquad (6.32)$$

where the A_j's and c_j's are complex quantities. To find the c_j's, we substitute Eqs. (6.30) and (6.32) into Eq. (6.24) to obtain

$$\frac{8\mathrm{Bu}}{\pi \mathrm{Bo}} (c_j^2 - b_T)(c_j^2 - 1) + i(c_j^2 - b_S)\left(c_j^2 - \left[\frac{3\mathrm{Bu}^2}{4\pi^2} + 1 \right] \right) = 0 \ , \quad (6.33)$$

where $b_S \equiv 1 - M_{S_\infty}^2$, $b_T \equiv 1 - M_{T_\infty}^2$ and $\mathrm{Bu} \equiv \alpha_\infty \ell$. The quantity Bu is called the Bouguer number. Equation (6.33) is a fourth-order equation for all finite, non-zero values of Bo and Bu. It remains fourth-order when Bo = 0, Bo → ∞, and Bu = 0. When Bu approaches infinity, however, Eq. (6.33) reduces to a second-order equation. The roots of c_j for the limiting cases of the parameters Bo, Bu and M_{S_∞} are as follows:

$$\mathrm{Bo} = 0, \qquad c_j^2 = b_T = 1 - M_{T_\infty}^2 \quad \text{and} \quad c_j^2 = 1 \ ,$$

$$\mathrm{Bo} \to \infty, \qquad c_j^2 = b_S = 1 - M_{S_\infty}^2 \quad \text{and} \quad c_j^2 = 1 + 3(\mathrm{Bu}/2\pi)^2,$$

$$\mathrm{Bu} = 0, \qquad c_j^2 = b_S = 1 - M_{S_\infty}^2 \quad \text{and} \quad c_j^2 = 1 \ , \qquad (6.34a)$$

$$\mathrm{Bu} \to \infty, \qquad c_j^2 = b_S = 1 - M_{S_\infty}^2 \quad \text{and} \quad c_j^2 \to \infty \ ,$$

$$M_{S_\infty} = 0, \qquad c_j^2 = 1 \ , \quad \text{and} \quad c_j^2 = \frac{\left[\left(\frac{8\mathrm{Bu}}{\pi \mathrm{Bo}}\right)^2 + \frac{3\mathrm{Bu}^2}{4\pi^2} + 1 \right] + i\left[\frac{48}{\mathrm{Bo}} \left(\frac{\mathrm{Bu}}{2\pi}\right)^3 \right]}{\left(\frac{8\mathrm{Bu}}{\pi \mathrm{Bo}}\right)^2 + 1} \ .$$

It is also of interest to note that for $\gamma = 1$

$$c_j^2 = 1 - M_{S_\infty}^2 \quad \text{and} \quad c_j^2 = \frac{\left[(\frac{8Bu}{\pi Bo})^2 + \frac{3Bu^2}{4\pi^2} + 1 \right] + i \left[\frac{48}{Bo} (\frac{Bu}{2\pi})^3 \right]}{(\frac{8Bu}{\pi Bo})^2 + 1} \; .$$

In the general case, since Eq. (6.33) is a quadratic equation in c_j^2, half of the roots will have positive real part. In view of the boundary condition (6.3), these roots are inadmissible. The roots with negative real part can be found by solving Eq. (6.33) formally to obtain

$$c_{1,2} = - \left[\frac{\frac{8Bu}{\pi Bo} (1+b_T) + i \left(\frac{3Bu^2}{4\pi^2} + 1 + b_S \right)}{2(\frac{8Bu}{\pi Bo} + i)} \right.$$

$$\left. \mp \frac{\sqrt{\left\{ \frac{8Bu}{\pi Bo} (1+b_T) + i \left(\frac{3Bu^2}{4\pi^2} + 1 + b_S \right) \right\}^2 - 4 \left(\frac{8Bu}{\pi Bo} + i \right) \left[\frac{8Bu}{\pi Bo} b_T + i b_S \left(\frac{3Bu^2}{4\pi^2} + 1 \right) \right]}}{2(\frac{8Bu}{\pi Bo} + i)} \right]^{\frac{1}{2}} .$$

(6.34b)

The A_j's in Eq. (6.32) can be found by imposing the boundary conditions as follows: Substituting Eqs. (6.30) and (6.32) into boundary condition (6.25), we have

$$\sum_{j=1}^{2} A_j \, c_j = \frac{\epsilon}{\ell} 2\pi \; . \qquad (6.35)$$

Substituting Eqs. (6.30) and (6.32) into boundary condition (6.29) and making use of Eq. (6.33), we also obtain

$$\sum_{j=1}^{2} \frac{Bu}{2\pi} \frac{A_j(c_j^2 - b_T)\,(2c_j - 3\,\frac{Bu}{2\pi})}{\left[c_j - 3(\frac{Bu}{2\pi})^2 - 1\right]} = D \ . \qquad (6.36)$$

To simplify the analysis we now assume $D = 0$ (i.e., the wall temperature is constant and is the same as that at infinity). Solving the simultaneous Eqs. (6.35) and (6.36) with $D = 0$, we find

$$\frac{A_1}{2\pi(\frac{\epsilon}{\ell})} = \frac{1}{c_1 - \dfrac{c_2(c_1^2 - b_T)\,(2c_1 - 3\,\frac{Bu}{2\pi})\left[c_2^2 - \left(3\,\frac{Bu^2}{4\pi^2} + 1\right)\right]}{(c_2^2 - b_T)\,(2c_2 - 3\,\frac{Bu}{2\pi})\left[c_1^2 - \left(3\,\frac{Bu^2}{4\pi^2} + 1\right)\right]}}, \qquad (6.37a)$$

$$\frac{A_2}{2\pi\,(\frac{\epsilon}{\ell})} = \frac{1}{c_2 - \dfrac{c_1(c_2^2 - b_T)\,(2c_2 - 3\,\frac{Bu}{2\pi})\left[c_1^2 - \left(3\,\frac{Bu^2}{4\pi^2} + 1\right)\right]}{(c_1^2 - b_T)\,(2c_1 - 3\,\frac{Bu}{2\pi})\left[c_2^2 - \left(3\,\frac{Bu^2}{4\pi^2} + 1\right)\right]}} \ . \qquad (6.37b)$$

It is convenient to introduce the real and imaginary parts of c_j such that $c_j = -(\delta_j + i\lambda_j)$, where δ_j and λ_j are positive quantities. The velocity potential Eq. (6.30) as a function of x_1 and x_2 is then given by

$$\phi(x_1, x_2) = \frac{U_\infty \ell}{2\pi} \operatorname{Re} \left\{ \sum_{j=1}^{2} A_j \exp\left[\frac{2\pi}{\ell} (c_j x_2 + i x_1) \right] \right\}$$

$$= U_\infty \epsilon \sum_{j=1}^{2} e^{-2\pi\delta_j (x_2/\ell)} \tag{6.38}$$

$$\times \left[a_j \cos 2\pi \left(\frac{x_1 - \lambda_j x_2}{\ell} \right) - b_j \sin 2\pi \left(\frac{x_1 - \lambda_j x_2}{\ell} \right) \right] ,$$

where a_j and b_j are the real and imaginary parts of $A_j/(2\pi\epsilon/\ell)$. Equation (6.38) represents, in general, two systems of waves with different amplitude, damping factor, and direction of propagation. Each of these waves, with amplitude A_j, decays exponentially with x_2 with a damping factor δ_j and along straight lines with slope (measured from the positive x_2-coordinate) of

$$\frac{dx_1}{dx_2} = \lambda_j \quad . \tag{6.39}$$

The variation of δ_j, λ_j and A_j as functions of the parameters M_{S_∞}, Bu, and [illegible] will be examined in detail below. We shall see that the A_1-term differs only slightly from the classical solution, whereas the A_2-term has no counterpart in classical theory. Following Vincenti and Baldwin (1962), we refer to the A_1- and A_2-terms as the modified classical wave and the radiation-induced wave, respectively.

The disturbed velocities are given by

$$\frac{u_1'}{U_\infty} = \frac{1}{U_\infty}\frac{\partial\phi}{\partial x_1} = -\frac{2\pi\epsilon}{\ell}\sum_{j=1}^{2}\sqrt{a_j^2 + b_j^2}\; e^{-2\pi\delta_j(x_2/\ell)} \sin 2\pi \frac{(x_1+\Delta_j x_1)-\lambda_j x_2}{\ell} , \tag{6.40a}$$

and

$$\frac{u_2'}{U_\infty} = \frac{1}{U_\infty}\frac{\partial\phi}{\partial x_2} = -\frac{2\pi\epsilon}{\ell}\sum_{j=1}^{2}(a_j\delta_j + b_j\lambda_j)\; e^{-2\pi\delta_j(x_2/\ell)} \cos 2\pi\left(\frac{x_1-\lambda_j x_2}{\ell}\right) , \tag{6.40b}$$

where

$$\frac{\Delta_j x_1}{\ell} = \frac{1}{2\pi}\tan^{-1}\left(\frac{b_j}{a_j}\right) .$$

Within the framework of the linearized theory, the pressure coefficient on the wall is

$$(C_p)_{x_2=0} = \left(-\frac{2u_1'}{U_\infty}\right)_{x_2=0} = \frac{4\pi\epsilon}{\ell}\sum_{j=1}^{2}\sqrt{a_j^2+b_j^2}\,\sin 2\pi\left(\frac{x_1+\Delta_j x_1}{\ell}\right) . \tag{6.41}$$

Equations (6.40a) and (6.41) show that both the horizontal disturbed velocity and the pressure distribution at the wall are out of phase with the wall, while Eq. (6.40b) shows that the vertical disturbed velocity is in phase.

The drag coefficient per unit wave length of the wall can be found from

$$C_d = \int_0^1 (C_p)_{x_2=0} \frac{dX_2}{dx_1} d(\frac{x_1}{\ell}) \quad . \tag{6.42}$$

This leads to

$$C_d = (\frac{2\pi\epsilon}{\ell})^2 \sum_{j=1}^{2} b_j \quad . \tag{6.43}$$

For the limiting cases of a transparent gas (Bu = 0), an opaque gas (Bu $\rightarrow \infty$), or a completely cold gas (Bo $\rightarrow \infty$), Eqs. (6.37) give

$$\frac{A_1}{2\pi(\frac{\epsilon}{\ell})} = \frac{1}{c_1} = - \frac{1}{(\delta_1 + i\lambda_1)} \quad , \tag{6.44a}$$

and

$$\frac{A_2}{2\pi(\frac{\epsilon}{\ell})} = 0 \quad . \tag{6.44b}$$

Equations (6.40), (6.41) and (6.43) give, for subsonic flow

$$\frac{u_1'}{U_\infty} = \frac{2\pi \frac{\epsilon}{\ell}}{\sqrt{1 - M_{S_\infty}^2}} \exp\left[- 2\pi \sqrt{1 - M_{S_\infty}^2} \frac{x_2}{\ell}\right] \sin 2\pi \frac{x_1}{\ell} \quad , \tag{6.45a}$$

$$\frac{u_2'}{U_\infty} = 2\pi \frac{\epsilon}{\ell} \exp\left[- 2\pi \sqrt{1 - M_{S_\infty}^2} \frac{x_2}{\ell}\right] \cos 2\pi \frac{x_1}{\ell} \quad , \tag{6.45b}$$

$$(C_p)_{x_2=0} = - \frac{4\pi(\frac{\epsilon}{\ell})}{\sqrt{1 - M_{S_\infty}^2}} \sin 2\pi \frac{x_1}{\ell} , \tag{6.45c}$$

$$C_d = 0 , \tag{6.45d}$$

and for supersonic flow

$$\frac{u_1'}{U_\infty} = - \frac{2\pi(\frac{\epsilon}{\ell})}{\sqrt{M_{S_\infty}^2 - 1}} \cos 2\pi \frac{x_1 - \sqrt{M_{S_\infty}^2 - 1}\, x_2}{\ell} , \tag{6.45e}$$

$$\frac{u_2'}{U_\infty} = 2\pi(\frac{\epsilon}{\ell}) \cos 2\pi \left(\frac{x_1 - \sqrt{M_{S_\infty}^2 - 1}\, x_2}{\ell} \right), \tag{6.45f}$$

$$(C_p)_{x_2=0} = \frac{4\pi(\frac{\epsilon}{\ell})}{\sqrt{M_{S_\infty}^2 - 1}} \cos 2\pi \frac{x_1}{\ell} , \tag{6.45g}$$

$$C_d = (2\pi \frac{\epsilon}{\ell})^2 \frac{U_\infty}{\sqrt{M_{S_\infty}^2 - 1}} . \tag{6.45h}$$

Equations (6.44) and (6.45) are identical to Ackeret's classical solution (1928) for flow over a wavy wall. For the limiting case of an infinitely hot gas (Bo = 0), the results are the same as given by Eqs. (6.44) and (6.45) but with the isentropic Mach number replaced by the isothermal Mach number. For this problem we thus see that the

radiation-induced wave disappears in the four limiting cases. In these limiting cases, the decay of the disturbance velocities is zero for supersonic flow. The drag coefficient and the clockwise rotation (from the positive x_2-coordinate) of the lines along which the exponential decay of the disturbance velocities takes place are both zero for subsonic flow.

Figure 2 through Fig. 9 show the variation of δ_j, λ_j, A_j and C_d as functions of the parameters M_{S_∞}, Bu, and Bo for $\gamma = 7/5$. Figure 2 shows the variation of δ_1 and λ_1 over the range of Mach number M_{S_∞} from 0 to 2.0 for specific values of Bu and Bo. The solid lines are values of δ_1 and λ_1 for the three limiting cases of a completely cold gas, a transparent gas, and an opaque gas. These are also corresponding to Ackeret's classical solution. For intermediate values of Bu and Bo, the damping factor δ_1 is never zero over any part of the Mach-number range. As the Mach number increases, δ_1 at first decreases slowly from the value of 1.0 at $M_{S_\infty} = 0$ (incompressible flow), decreases rapidly in the near-sonic region, and then varies only slightly in the supersonic region. The value of λ_1, which is a measure of the clockwise rotation of the lines along which the exponential decay takes place, is zero for incompressible flow. Its value is small in the subsonic region for all values of the Bouguer and Boltzmann numbers. It begins to grow rapidly in the near-sonic region and continues to grow in the supersonic region.

Figures 2a and 2b show the variation of δ_1 and λ_1 versus M_{S_∞} at discrete values of Bo for the thin- and thick-gas respectively. As the Boltzmann number increases, the curves of δ_1 and λ_1 shift to

the right toward the classical curves. By comparing these two figures, we note that a thin gas (Bu = 0.1) with Bo = 1.0 behaves essentially like a completely cold gas; whereas a thick gas (Bu = 10.0) with the same Boltzmann number behaves like an infinitely hot gas. This is so because for a gas at moderate Boltzmann numbers, if the absorption coefficient is small as in the thin-gas case, the effect of radiation is relatively unimportant. On the other hand, if its absorption coefficient is large, the effect of radiation becomes important.

Figures 2c and 2d show the variation of δ_1 and λ_1 versus M_{S_∞} at discrete values of Bu for a hot and cold gas respectively. As Bu increases, the curves for δ_1 and λ_1 first move away from the classical curve; with further increase in Bu, they eventually move back toward the classical curves. In the limit of infinite Bouguer number, they again coincide with the classical curves. By comparing Figs. 2c and 2d, we note that for Bo = 0.1, except for the cases when the Bouguer number is extremely small or large, the gas behaves essentially like an infinitely hot gas (cf. Figs. 2a and 2b); whereas for Bo = 10.0, the gas behaves like a completely cold gas.

Figures 3a and 3b show the variation of δ_1 and λ_1 for the complete range of Bouguer number for specific values of the Boltzmann number and the Mach number. To emphasize the boundary-layer-like behavior of the results, the horizontal scale for the Bouguer number from 10.0 to ∞ has been arbitrarily chosen. Thus the results in this range of the Bouguer number are qualitative in the horizontal direction while remaining quantitative vertically. The boundary-layer-like behavior becomes pronounced only for certain combinations of parameters, namely,

for a hot gas with extremely small or large Bouguer number. For a fixed pair of values of Mach and Boltzmann numbers, the values of δ_1 and λ_1 are qualitatively symmetric with respect to Bouguer number.

Figures 3c and 3d show the variation of δ_1 and λ_1 for the complete range of Bo for specific values of Bu and M_{S_∞}. The horizontal scale for Bu in the range of 10.0 to ∞ is again arbitrarily chosen. For a fixed pair of values of M_{S_∞} and Bu, the values of δ_1 and λ_1 are asymmetric with respect to Boltzmann number. Again, the boundary-layer-like behavior is observed for small Boltzmann number with either very small or very large Bouguer numbers.

Figure 4 shows the variation of δ_2 and λ_2 over the Mach-number range of 0 to 2.0 for specific values of Boltzmann number and Bouguer number. We see that for the radiation-induced wave, both δ_2 and λ_2 are practically independent of Mach number. For all finite, non-zero values of Bo and Bu, the values of δ_2 and λ_2 never become zero. Thus the radiation-induced wave is always damped, and the clockwise rotation of the lines along which the exponential decay takes place is always non-zero for all Mach numbers.

Since the values of δ_2 and λ_2 are practically independent of Mach number, a plot of these quantities versus Bu at specific values of Bo (or versus Bo at specific values of Bu) for a specific Mach number will be qualitatively representative of all Mach numbers. Figures 5a and 5b are such plots for $M_{S_\infty} = 2.0$. Figure 5a shows that as Bu varies from 0 to ∞, the damping factor δ_2 increases monotonically from 1 to ∞ while λ_2 increases monotonically from 0 to ∞. Figure 5b shows that as Bo varies from 0 to ∞, the damping increases

slowly and then increases or decreases to a certain asymptotic value depending on the value of Bu. At the same time, λ_2 increases from zero to a maximum and then decreases to zero. The variation of λ_2 with respect to Bo is thus roughly symmetric.

The amplitudes of the modified classical wave and the radiation-induced wave depend on the boundary conditions. Figure 6 shows the variation of the amplitudes of the two waves for the Mach-number range from 0 to 2.0 for specific values of Bu and Bo. It is seen that the modified classical wave predominates at all values of M_{S_∞}, Bu and Bo. This is only true for this particular problem, and would not hold for other situations (for example, for a flat wall with a periodic temperature distribution). For the limiting cases of a transparent gas (Bu $\rightarrow$ 0), an opaque gas (Bu $\rightarrow \infty$), and a completely cold gas (Bo $\rightarrow \infty$), the amplitude of the radiation-induced wave becomes zero, and the amplitude of the modified classical wave is identical to the classical solution represented by the solid lines. It is seen that in the classical solution, the amplitude becomes infinite at $M_{S_\infty} = 1$. For the limiting case of an infinitely hot gas (Bo = 0), the amplitude of the radiation-induced wave is again zero, and the amplitude of the modified classical wave becomes infinite at $M_{T_\infty} = 1$. For finite, non-zero values of Bo and Bu, the amplitude of the modified classical wave never becomes infinite. For fixed values of Bu and Bo, with increasing Mach number the amplitude of the modified classical wave increases slowly until the near-sonic region is reached, where it rises sharply to a maximum value and then declines. In the supersonic region, $|A_1|/(2\pi\epsilon/\ell)$ decreases slowly with increasing M_{S_∞}.

Figures 7a and 7b are plots of $|A_j|/(2\pi\epsilon/\ell)$ versus Bu for specific values of Bo and M_{S_∞}. As Bu varies from 0 to ∞, $|A_1|/(2\pi\epsilon/\ell)$ for subsonic flow increases from a certain value to a maximum and then declines back to the original value. For supersonic flow, $|A_1|/(2\pi\epsilon/\ell)$ decreases from a certain value to a minimum and then increases back to the original value. The boundary-layer-like behavior is again observed for a hot gas with extremely small or large Bouguer number.

Figures 7c and 7d are plots of $|A_j|/(2\pi\epsilon/\ell)$ versus Bo at discrete values of Bu for subsonic and supersonic flows respectively. As Bo varies from 0 to ∞, the amplitude of the modified classical wave decreases for subsonic flow and increases for supersonic flow. The rate of decrease or increase is rapid for a gas with extremely small or large Bouguer number.

Figure 8 shows that at all non-zero, finite, Bouguer, Boltzmann and Mach numbers, the drag coefficient never becomes zero or infinite. For subsonic flow, it is small for all values of Bu and Bo. In the near-sonic region, it rises sharply to a maximum, and then declines sharply into the supersonic region where it continues to decline slowly.

Figures 9a and 9b show the variation of the drag parameter $C_d/(2\pi\epsilon/\ell)^2$ versus Bu at discrete values of Bo for subsonic and supersonic flows respectively. Again the boundary-layer-like behavior is observed for a hot gas with extremely small or large Bouguer number.

Figures 9c and 9d show the variation of $C_d/(2\pi\epsilon/\ell)^2$ as a function of Bo at discrete values of Bu for subsonic and supersonic flows respectively. With increasing Bo, this parameter increases from

zero to a maximum and decreases back to zero for subsonic flow. For supersonic flow, it increases monotonically from one non-zero value to another. The rate of changes of the parameter is rapid for a hot gas with extremely small or large Bouguer number.

VII. CONCLUDING REMARKS

The spherical-harmonic method has been used by nuclear physicists extensively in connection with the design of nuclear reactors in the past decade. It has been shown that the first approximation of the spherical-harmonic method is highly accurate for such neutron-transport problems.

The formulation of radiating gasdynamics with the spherical-harmonic (differential) approximation appears promising. Although we have no way, as yet, of assertaining the accuracy of the approximation for multi-dimensional flow, the corresponding approximation in one-dimensional problems is known to be extremely accurate. This is shown by the works of Pearson (1964) and Ferziger (1965). For the problem of the radiation-resisted shock wave, the exact numerical solution of Pearson compared extremely well with the solution by Heaslet and Baldwin (1963) on the basis of the exponential approximation, which in turn is equivalent to the differential approximation. For the problem of radiative transfer between parallel plates, Ferziger shows that the differential approximation is extremely accurate as compared with an

exact solution. Furthermore, the present formulation recovers the correct thin- and thick-gas approximations in the appropriate asymptotic situations. All of these facts indicate that this approach is at least qualitatively correct.

The grey-gas approximation is obviously unrealistic from a physical point of view. For the linearized theory, where temperatures and other physical properties do not change appreciably, this assumption may not, however, be critical.

From the mathematical point of view, the assumption of non-scattering is not essential to the differential approximation. Had we taken into consideration the effect of scattering, the coefficient in the linearized equations would be modified accordingly.

Since radiative transfer is an irreversible process, we expect that the effects of radiative nonequilibrium might be similar in some ways to those of chemical or vibrational nonequilibrium. The solution for radiative flow over a wavy wall does, in fact, show the occurrence of pressure drag at subsonic speeds and a smoothing of the transition from subsonic to supersonic speeds. In these respects, the results are indeed similar to the flow over a wavy wall with chemical or vibrational nonequilibrium (Vincenti (1959)).

APPENDIX A

RECURRENCE AND ORTHOGONALITY RELATIONS FOR THE SPHERICAL HARMONICS

In this section, we shall cite without proof the well-known formulas and relations of the spherical harmonics. For a detailed discussion of spherical harmonics, the reader is referred to standard texts on advanced mathematics (see for example, Sommerfeld (1964) or Irving and Mullineaux (1959)).

The spherical harmonic $Y_{\ell}^{m}(\bar{\Omega})$ is related to the associated Legendre function $P_{\ell}^{m}(\cos\theta)$ by

$$Y_{\ell}^{m}(\bar{\Omega}) \equiv C_{\ell}^{m}\, e^{im\phi}\, P_{\ell}^{m}(\mu) \ , \tag{A-1}$$

where the upper index m is positive, $\mu \equiv \cos\theta$, and $C_{\ell}^{m} = \left[\frac{(\ell-m)!}{(\ell+m)!}\right]^{1/2}$. If the upper index is negative, it is related to the complex conjugate $\bar{Y}_{\ell}^{m}(\bar{\Omega})$ by

$$Y_{\ell}^{-m}(\bar{\Omega}) \equiv (-1)^{m}\, C_{\ell}^{m}\, e^{-im\phi}\, P_{\ell}^{m}(\mu) = (-1)^{m}\, \bar{Y}_{\ell}^{m}(\bar{\Omega}). \tag{A-2}$$

It follows from (A-1) and (A-2) that

$$\frac{Y_{\ell}^{m}(\bar{\Omega}) + \bar{Y}_{\ell}^{m}(\bar{\Omega})}{2C_{\ell}^{m}} = \frac{Y_{\ell}^{m}(\bar{\Omega}) + (-1)^{m}\, Y_{\ell}^{-m}(\bar{\Omega})}{2C_{\ell}^{m}} = (\sin\theta)^{m} \cos m\phi\, \frac{d^{m}}{d\mu^{m}}\, P_{\ell}(\mu) \ , \tag{A-3a}$$

and

$$\frac{Y_{\ell}^{m}(\bar{\Omega}) - \bar{Y}_{\ell}^{m}(\bar{\Omega})}{2C_{\ell}^{m}} = \frac{Y_{\ell}^{m}(\bar{\Omega}) - (-1)^{m}\, Y_{\ell}^{-m}(\bar{\Omega})}{2C_{\ell}^{m}} = (\sin\theta)^{m} \sin m\phi\, \frac{d^{m}}{d\mu^{m}} P_{\ell}(\mu) \ . \tag{A-3b}$$

In view of Eqs.(A-1), (A-2), and (A-3), the direction cosines given by Eq. (2.18) are, in terms of the spherical harmonics,

$$\ell_2 = \cos\theta = Y_1^0(\bar{\Omega}) \ , \tag{A-4a}$$

$$\ell_1 = \sin\theta \sin\phi = \frac{Y_1^1(\bar{\Omega}) - \bar{Y}_1^1(\bar{\Omega})}{\sqrt{2}} = \frac{Y_1^1(\bar{\Omega}) + Y_1^{-1}(\bar{\Omega})}{\sqrt{2}} \ , \tag{A-4b}$$

$$\ell_3 = \sin\theta \cos\phi = \frac{Y_1^1(\bar{\Omega}) + \bar{Y}_1^1(\bar{\Omega})}{\sqrt{2}} = \frac{Y_1^1(\bar{\Omega}) - Y_1^{-1}(\bar{\Omega})}{\sqrt{2}} \ . \tag{A-4c}$$

The addition theorem for spherical harmonics is

$$P_{\ell}(\bar{\Omega}\cdot\bar{\Omega}') = \sum_{m=-\ell}^{\ell} Y_{\ell}^{m}(\bar{\Omega})\, \bar{Y}_{\ell}^{m}(\bar{\Omega}') \ . \tag{A-5}$$

The orthogonal property and recurrence relations for the spherical harmonics are as follows:

$$\int_{\Omega} \bar{Y}_{\ell'}^{m'}(\bar{\Omega})\, Y_{\ell}^{m}(\bar{\Omega})\, d\Omega = (-1)^{m} \int_{\Omega} Y_{\ell'}^{-m'}(\bar{\Omega})\, Y_{\ell}^{m}(\bar{\Omega})\, d\Omega = \frac{4\pi}{(2\ell+1)}\, \delta_{\ell\ell'}\, \delta_{mm'} \ , \tag{A-6}$$

$$\int_{\Omega} P_{\ell'}^{m'}(\mu)\, P_{\ell}^{m}(\mu)\, d\Omega = \frac{4\pi}{(2\ell+1)}\, \frac{(\ell+m)!}{(\ell-m)!}\, \delta_{\ell\ell'}\, \delta_{mm'} \ , \tag{A-7}$$

$$\mu\, P_{\ell}^{m}(\mu) = \frac{(\ell+m)\, P_{\ell-1}^{m}(\mu) + (\ell+1-m)\, P_{\ell}^{m}(\mu)}{(2\ell+1)} \ , \tag{A-8}$$

$$P_{\ell}^{m}(\mu) = \frac{P_{\ell+1}^{m+1}(\mu) - P_{\ell-1}^{m+1}(\mu)}{(2\ell+1)\sin\theta} \ , \tag{A-9}$$

$$P_{\ell}^{m}(\mu) = \frac{1}{(2\ell+1)\sin\theta}\left[(\ell+2-m)(m-1-\ell)\, P_{\ell+1}^{m-1}(\mu) + (\ell+m-1)(\ell+m)\, P_{\ell-1}^{m-1}(\mu)\right]. \tag{A-10}$$

APPENDIX B

DETAILED DERIVATION OF EQUATION (4.4)

The radiation-transport equation for a grey gas in quasi-equilibrium is

$$\frac{1}{c}\frac{\partial I}{\partial t} + \bar{\Omega}\cdot \mathbf{grad}\ I = -(\alpha+\eta)\ I + \frac{\alpha\sigma T^4}{\pi} + \eta\int_{\Omega'} I\kappa\, d\Omega' \ . \tag{B-1}$$

Following the procedure used in neutron-transport theory, we first expand the radiation intensity in a series of spherical harmonics as follows:

$$I(\bar{r}, \bar{\Omega}, t) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} A_\ell^m(\bar{r}, t)\, Y_\ell^m(\bar{\Omega}) \ , \qquad \text{(B-2)}$$

where the $A_\ell^m(\bar{r},t)$'s are functions to be found and the $Y_\ell^m(\bar{\Omega})$'s are spherical harmonics. For reasons explained in Section 4.1, the scattering function can be expanded in a series in terms of the Legendre polynomials $P_\ell(\cos\theta_o)$. Utilizing the addition theorem (A-5), we can therefore write

$$\kappa(\mu_o, \bar{\Omega}') = \sum_{\ell=0}^{\infty} \kappa_\ell\, P_\ell(\mu_o) = \sum_{\ell=0}^{\infty} \kappa_\ell\, P_\ell(\bar{\Omega}\cdot\bar{\Omega}') = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \kappa_\ell\, Y_\ell^m(\bar{\Omega})\, \bar{Y}_\ell^m(\bar{\Omega}') .$$

(B-3)

Substituting Eqs. (B-2) and (B-3) into Eq. (B-1), we have

$$\left(\frac{1}{c}\frac{\partial}{\partial t} + \bar{\Omega}\cdot\nabla + \alpha + \eta\right) \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} A_\ell^m(\bar{r},t)\, Y_\ell^m(\bar{\Omega}) - \frac{\alpha\sigma T^4}{\pi}$$

$$= \eta \int_{\Omega'} \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} A_\ell^m(\bar{r},t)\, Y_\ell^m(\bar{\Omega}') \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \kappa_\ell\, Y_\ell^m(\bar{\Omega})\, \bar{Y}_\ell^m(\bar{\Omega}')\, d\Omega' .$$

(B-4)

Using the orthogonality relation (A-6), we can simplify the integral term in Eq. (B-4) to

$$\eta \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} A_\ell^m(\bar{r},t) \sum_{\ell=0}^{\infty} \kappa_\ell \sum_{m=-\ell}^{\ell} Y_\ell^m(\bar{\Omega}) \int_{\Omega'} Y_\ell^m(\bar{\Omega}')\, \bar{Y}_\ell^m(\bar{\Omega}')\, d\Omega'$$

$$= \eta \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} A_\ell^m(\bar{r},t)\, \kappa_\ell\, Y_\ell^m(\bar{\Omega}) \left(\frac{4\pi}{2\ell+1}\right) . \qquad \text{(B-5)}$$

With this result, Eq. (B-4) can be rearranged to yield

$$\sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} Y_{\ell}^{m}(\bar{\Omega}) \left\{ \left[\frac{1}{c} \frac{\partial}{\partial t} + \bar{\Omega} \cdot \nabla + (\alpha+\eta) - \eta \frac{\kappa_{\ell} 4\pi}{2\ell+1} \right] A_{\ell}^{m}(\bar{r},t) - \frac{\alpha\sigma T^4}{\pi} \delta_{0\ell} \delta_{0m} \right\} = 0.$$

(B-6)

To eliminate the spherical harmonics in Eq. (B-6), we multiply Eq. (B-6) by $\bar{Y}_{\ell}^{m}(\bar{\Omega})$ and then integrate over the whole range of solid angle. The first, third, fourth, and fifth terms in Eq. (B-6) immediately yield

$$\frac{1}{c} \frac{\partial A_{\ell}^{m}}{\partial t} + (\alpha+\eta) A_{\ell}^{m} - \eta \frac{\kappa_{\ell} 4\pi}{(2\ell+1)} A_{\ell}^{m} - \frac{\alpha\sigma T^4}{\pi} \delta_{0\ell} \delta_{0m} \quad . \tag{B-7}$$

The second term in Eq. (B-6) needs more attention. We first note that

$$\bar{\Omega} \cdot \nabla = \sin\theta \sin\phi \frac{\partial}{\partial x_1} + \cos\theta \frac{\partial}{\partial x_2} + \sin\theta \cos\phi \frac{\partial}{\partial x_3} \, .$$

(B-8)

If the second term in Eq. (B-6) is multiplied by $\bar{Y}_{\ell}^{m}(\bar{\Omega})$ and then integrated over the whole range of solid angle, it becomes, after the substitution of Eq. (B-8),

$$\sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \int_{\Omega} Y_{\ell}^{m}(\bar{\Omega}) \, \bar{Y}_{\ell'}^{m'}(\bar{\Omega}) \tag{B-9}$$

$$\left[\sin\theta \sin\phi \frac{\partial}{\partial x_1} + \cos\theta \frac{\partial}{\partial x_2} + \sin\theta \cos\phi \frac{\partial}{\partial x_3} \right] A_{\ell}^{m} \, .$$

Consider now the first term in Eq. (B-9). With the aid of Eqs. (A-1) and (A-2), and after replacing $\sin \phi$ by its exponential form, we have

$$- \frac{1}{2} \frac{\partial A_{\ell}^{m}}{\partial x_1} C_{\ell}^{m} C_{\ell'}^{m'}$$

$$\times \left[\int_{\Omega} \sin \theta \, e^{i(m-m'+1)\phi} P_{\ell}^{m}(\mu) \, P_{\ell'}^{m'}(\mu) \, d\Omega + \int_{\Omega} \sin \theta \, e^{i(m-m'-1)\phi} P_{\ell}^{m}(\mu) P_{\ell'}^{m'}(\mu) d\Omega \right] . \tag{B-10}$$

Making use of Eq. (A-9) and then utilizing Eq. (A-7), we obtain the first term in Eq. (B-10) as

$$- i2\pi \frac{[(\ell+m)(\ell+m-1)]^{1/2}}{(2\ell-1)(2\ell+1)} \frac{\partial A_{\ell-1}^{m-1}}{\partial x_1} + i2\pi \frac{[(\ell-m+2)(\ell-m+1)]^{1/2}}{(2\ell+3)(2\ell+1)} \frac{\partial A_{\ell+1}^{m-1}}{\partial x_1} .$$

Similarly, using Eq. (A-10) and Eq. (A-7), we obtain the second term in Eq. (B-10) as

$$- i2\pi \frac{[(\ell-m)(\ell-m-1)]^{1/2}}{(2\ell-1)(2\ell+1)} \frac{\partial A_{\ell-1}^{m+1}}{\partial x_1} + i2\pi \frac{[(\ell+m+2)(\ell+m+1)]^{1/2}}{(2\ell+3)(2\ell+1)} \frac{\partial A_{\ell+1}^{m+1}}{\partial x_1} .$$

The remaining terms in Eq. (B-9) can be handled in a similar manner. Collecting these results and those of Eq. (B-7), we obtain Eq. (4.4).

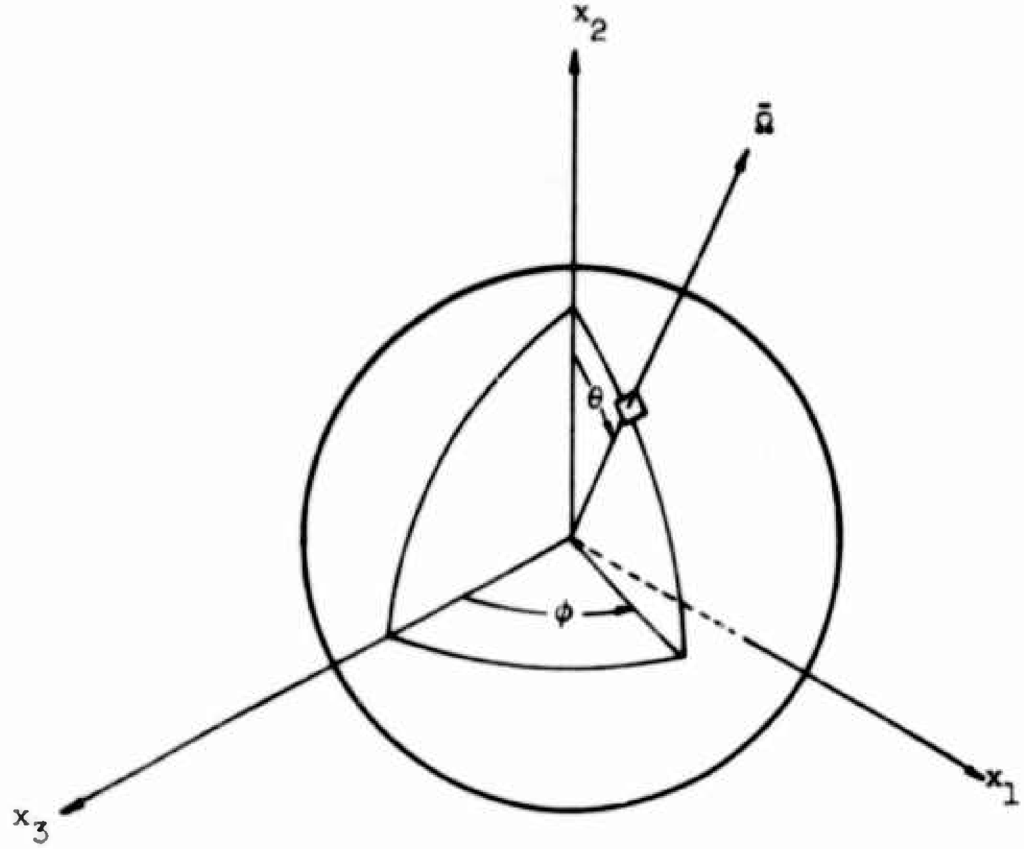

Figure 1. Coordinate System.

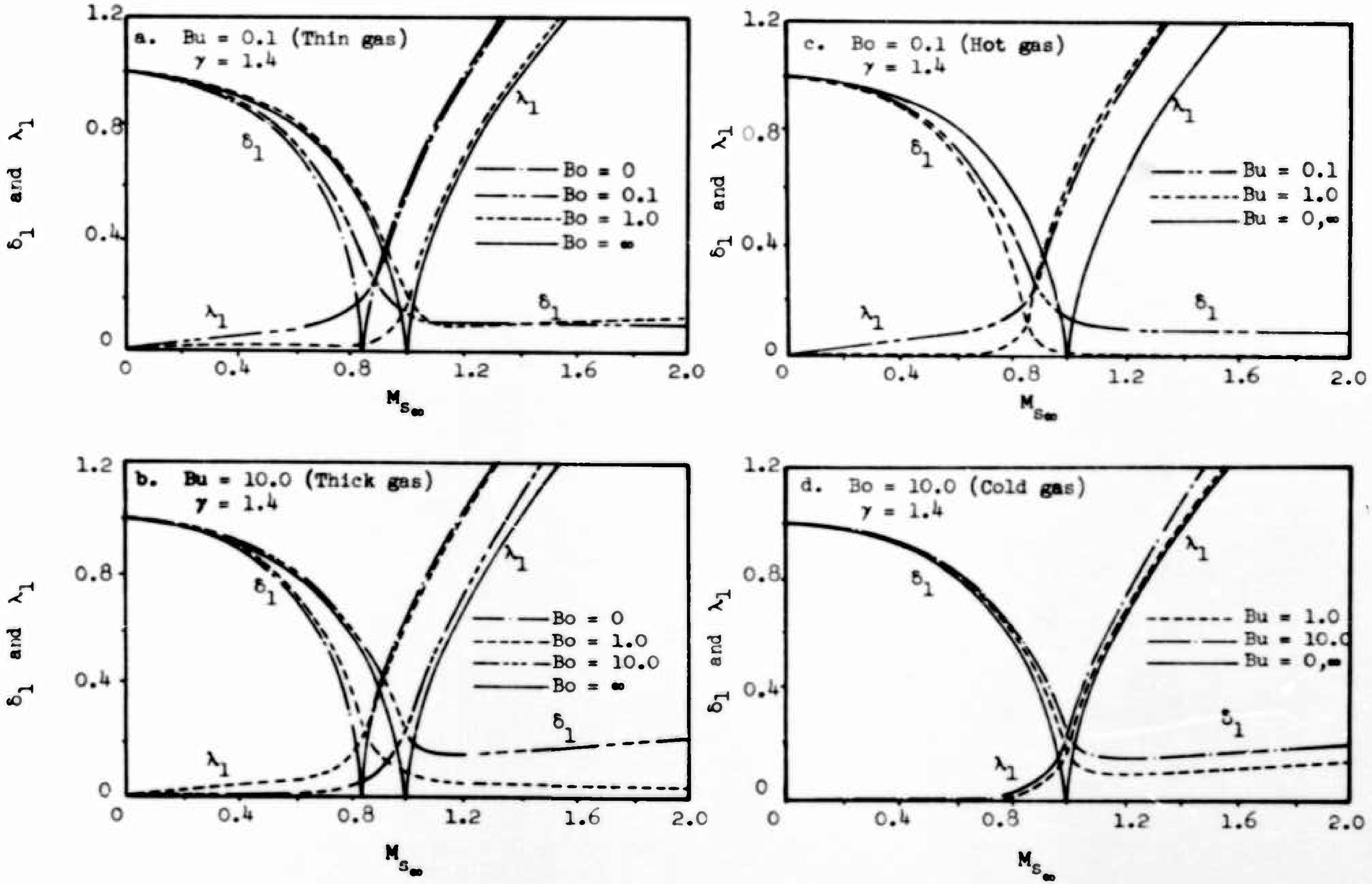

Figure 2. Values of δ_1 and λ_1 versus M_{S_∞} for specific values of Bu and Bo .

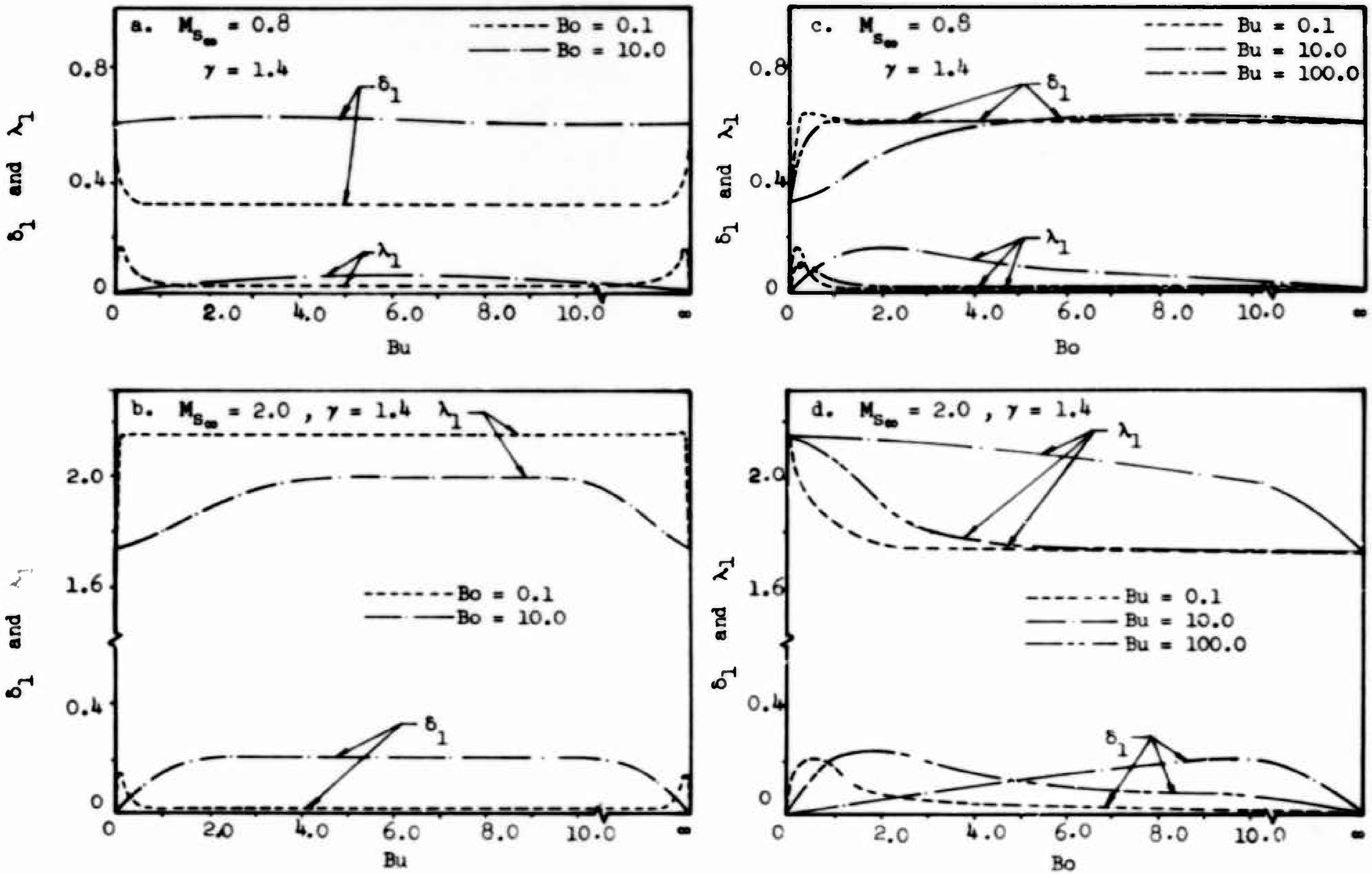

Figure 3. Values of δ_1 and λ_1 as functions of Bu and Bo at subsonic and supersonic speeds.

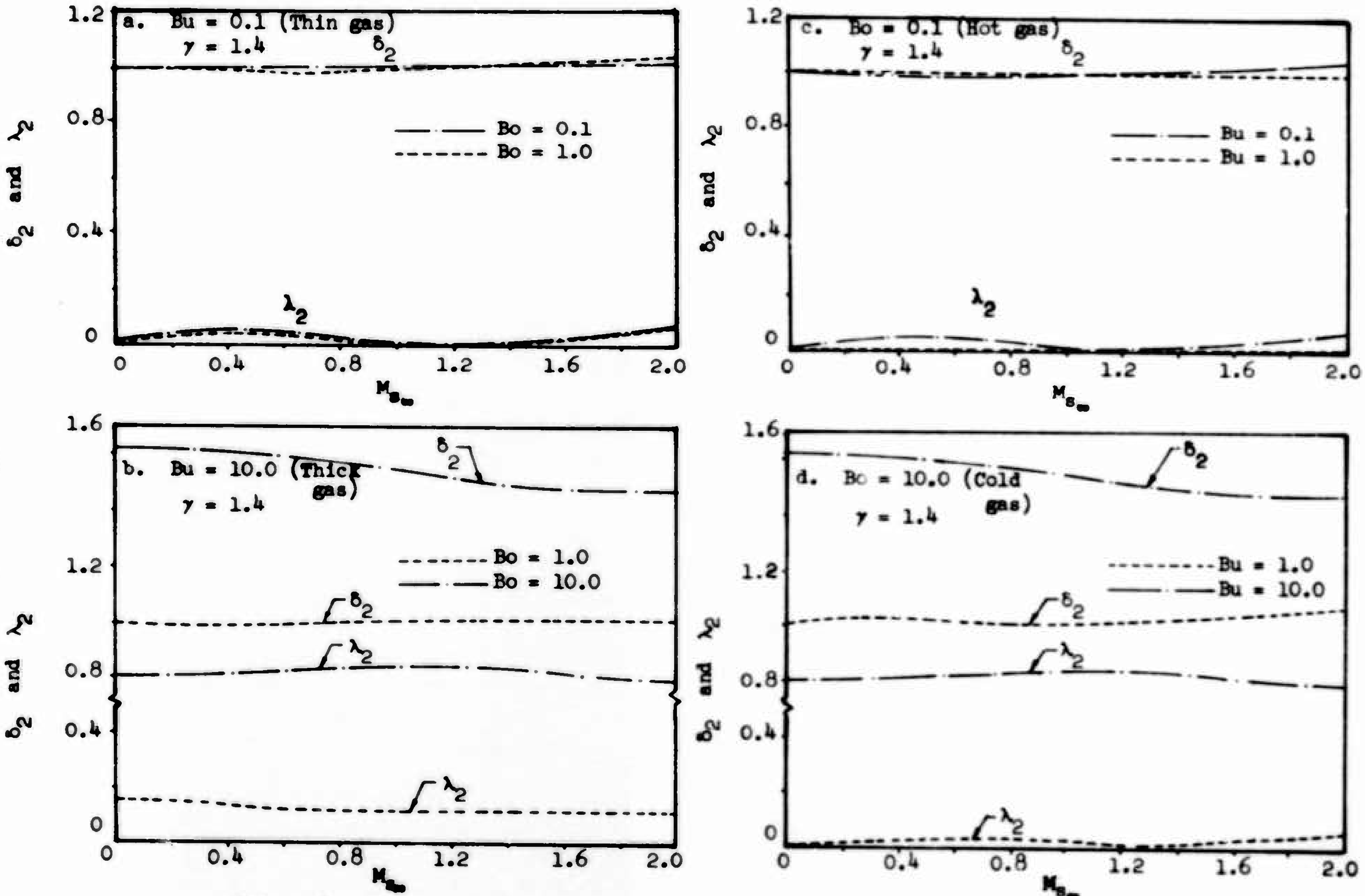

Figure 4. Values of δ_2 and λ_2 versus M_{s_∞} for specific values of Bu and Bo .

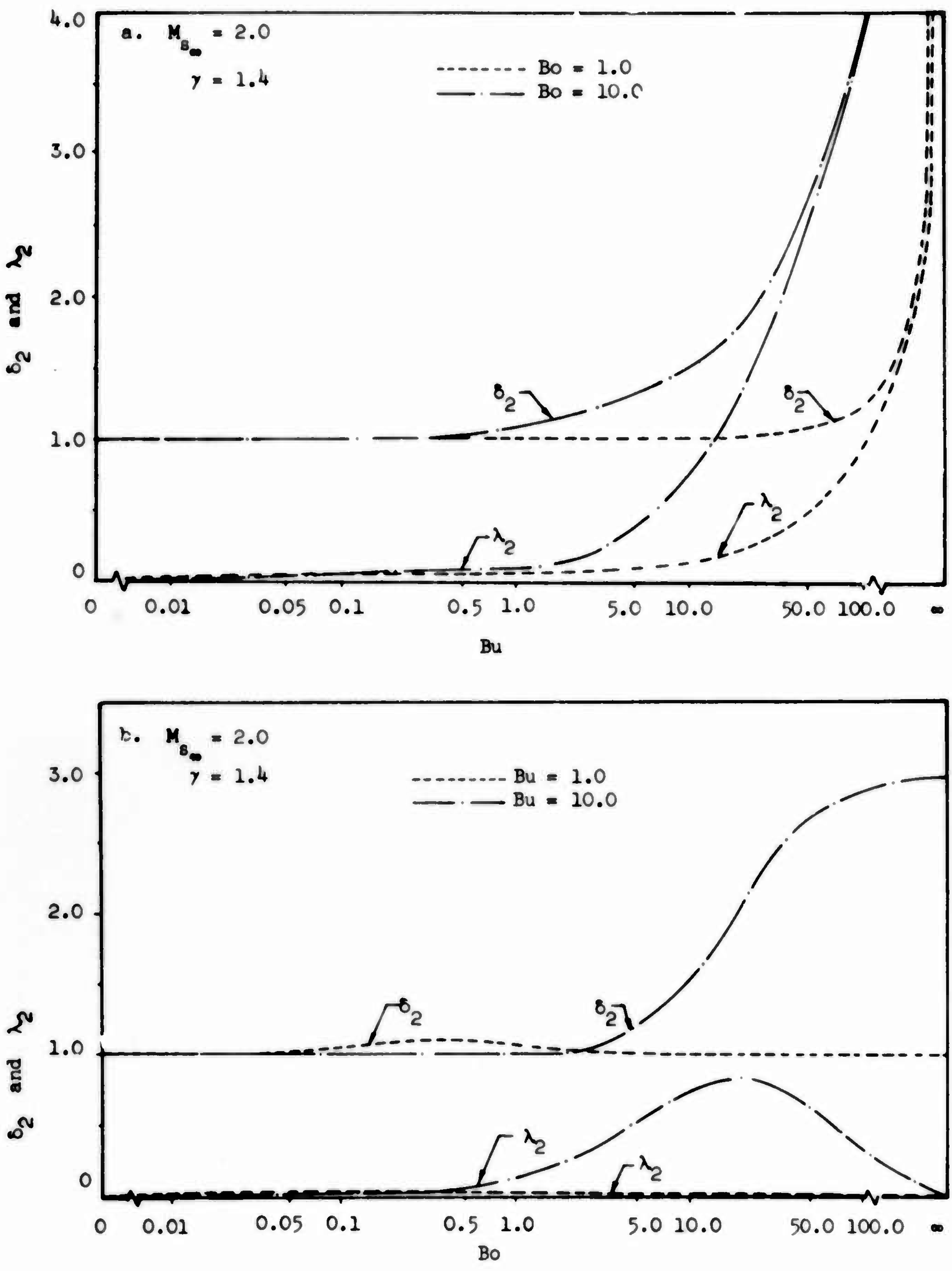

Figure 5. Values of δ_2 and λ_2 as functions of Bu and Bo for $M_{s_\infty} = 2.0$.

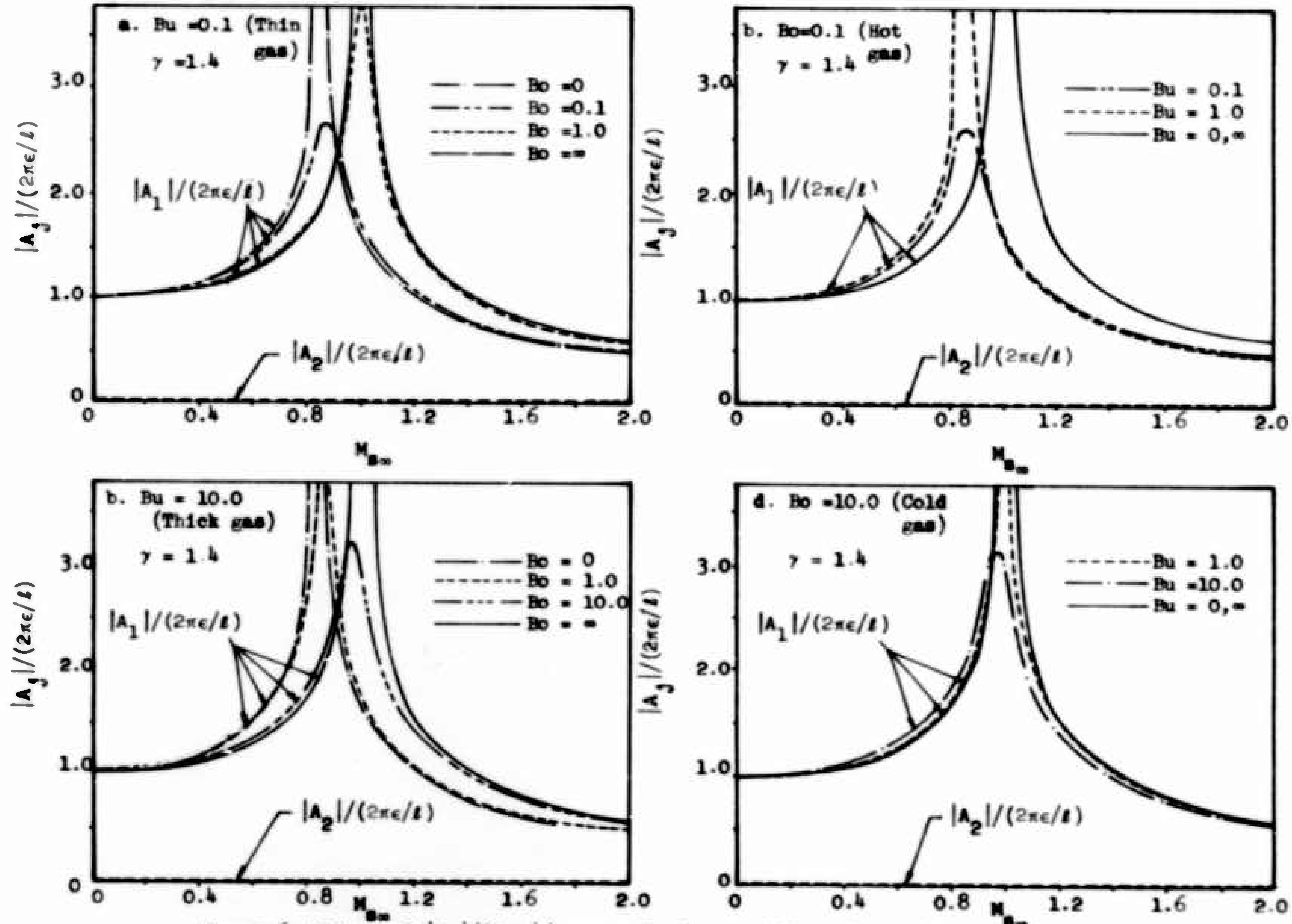

Figure 6. Values of $|A_j|/(2\pi\epsilon/\ell)$ versus $M_{S\infty}$ for specific values of Bu and Bo.

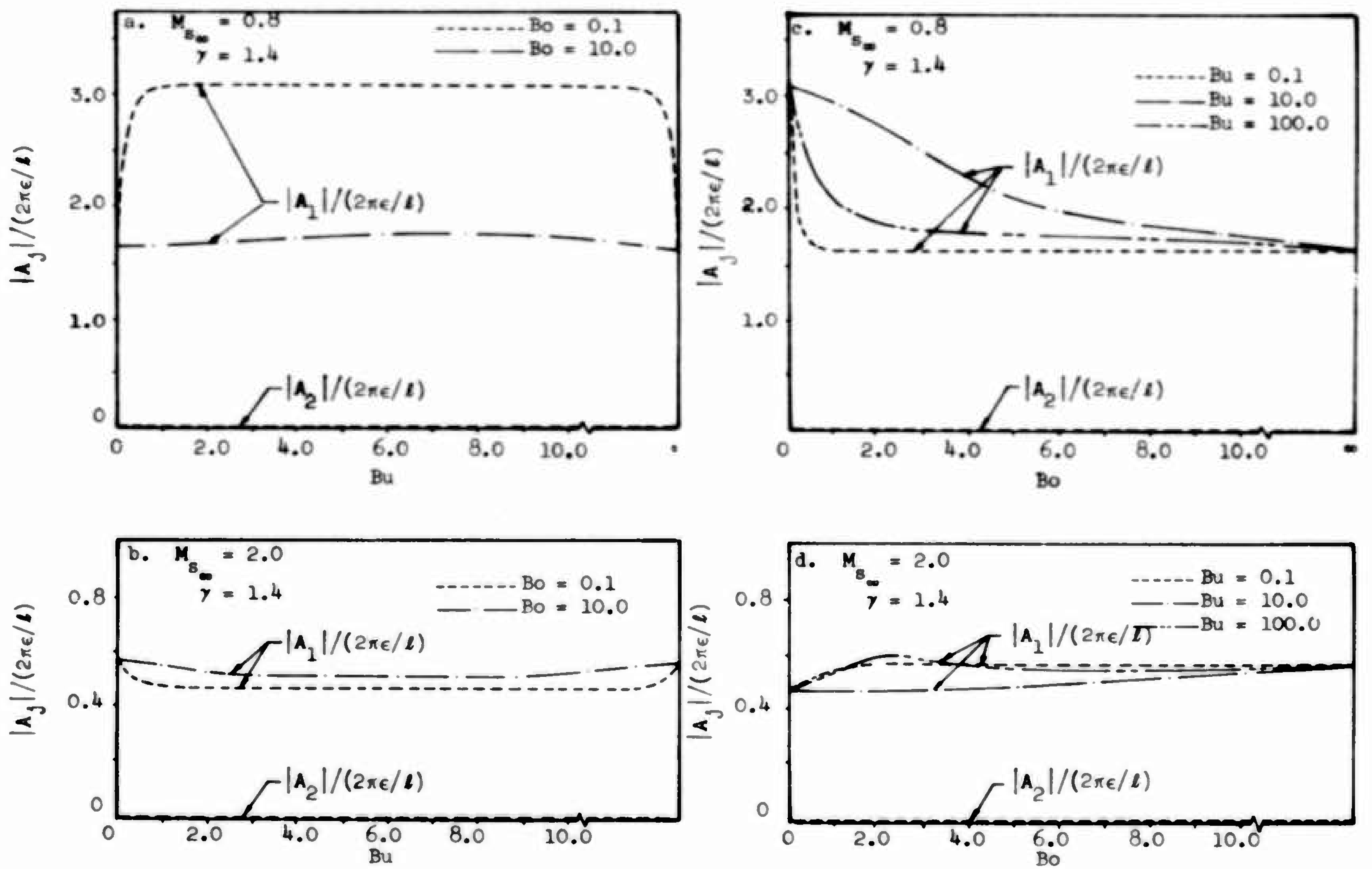

Figure 7. Values of $|A_j|/(2\pi\epsilon/\ell)$ as functions of Bu and Bo at subsonic and supersonic speeds.

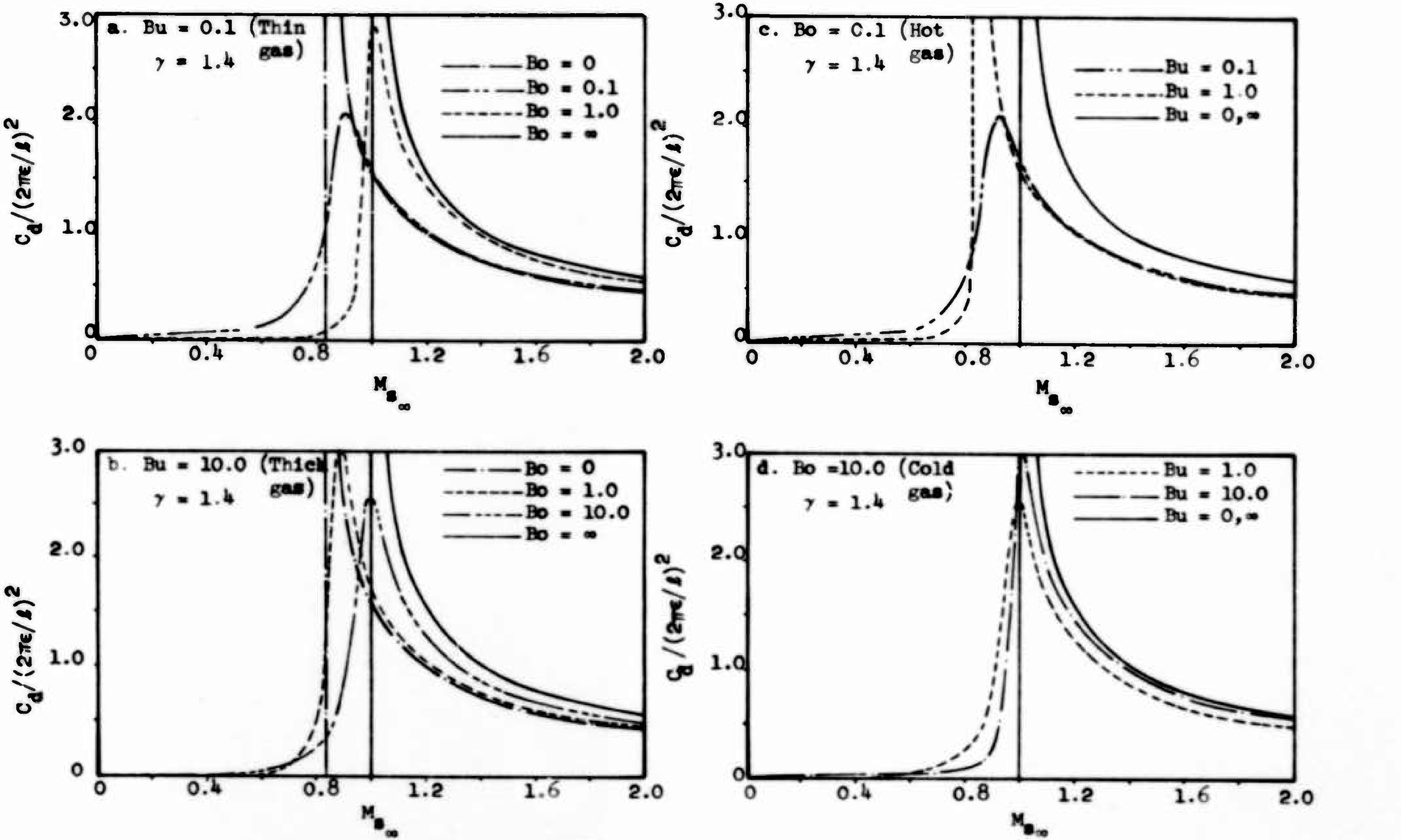

Figure 8. Values of $C_d/(2\pi\epsilon/\ell)^2$ versus M_{s_∞} for specific values of Bu and Bo .

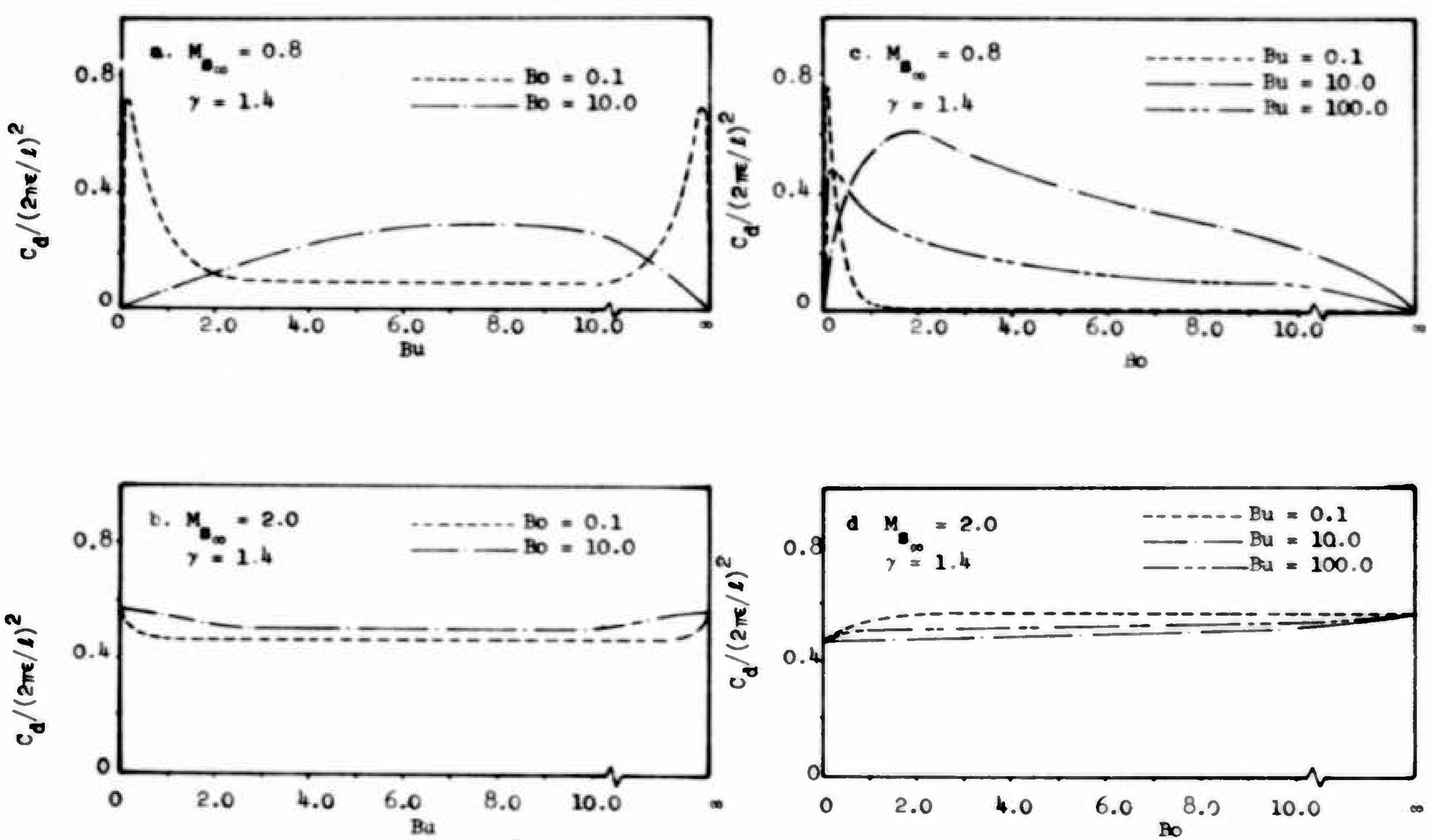

Figure 9. Values of $C_d/(2\pi\epsilon/\ell)^2$ as functions of Bu and Bo at subsonic and supersonic speeds.

REFERENCES

Ackeret, J., 1928, Über Luftkräfte bei sehr grossen Geschwindigkeiten insbesondere bei ebenen Strömungen, Helv. Phys. Acta 1, 301-322.

Chandrasekhar, S. 1944, On the radiative equilibrium of a stellar atomsphere, Astrophysical J. 99, 180-190.

Chandrasekhar, S. 1960, Radiative Transfer, Dover Publications, Inc., New York.

Cheng, P. 1964, Two-dimensional radiating gas flow by a moment method, AIAA J. 2, 1662-1664.

Chu, B. T. 1957, Wave propagation and the method of characteristics in radiating gas mixtures with applications to hypersonic flow. WADC TN-57-213.

Davison, B. 1956, Neutron Transport Theory, Oxford University Press, London, Ch. V. p. 97.

Einstein, A. 1905, Über einen die Erzeugung und Verwandlung des Lichtes betreffenden heuristischen Gesichtspunkt, Ann. d. Phys. 17, 132-148.

Ferziger, J. H. and Simmons, G. M. 1965, Radiative transfer between parallel plates. To appear.

Goulard, R. and Goulard, M. 1960, One-dimensional energy transfer in radiant media, Int. J. Heat Mass Transfer, Vol. 1, p. 81.

Goulard, R. 1962, Fundamental equations of radiation gasdynamics, Purdue Univ., School of Aeronautical and Engineering Sciences, Lafayette, Indiana, Rept. No. A and ES 62-4.

Grad, H. 1949, On the kinetic theory of rarefied gases, Comm. Pure and Appl. Math., 2, 331-407.

Heaslet, M. A. and Baldwin, B. S. 1963, Predictions of the structure of radiation-shock waves, The Physics of Fluid, Vol. 6, No. 6, 781-791.

Irving, J. and Mullineaux, N. 1959, Mathematics in Physics and Engineering, Academic Press, New York.

Kourganoff, V. 1952, Basic Methods in Transfer Problems, Oxford Univ. Press.

Lick, W. J. 1964, The propagation of small disturbances in a radiating gas. J. Fluid Mech. 18, 274-285.

Lighthill, M. J. 1960, Dynamics of a dissociating gas, Part 2. Quasi-equilibrium transfer theory, J. Fluid Mech. 8, 161-182.

Marshak, R. E. 1947, Note on the spherical harmonic method as applied to the Mile problem for a sphere, Phys. Rev. 71, 443-446.

Mark, J. C. 1944, The spherical harmonic method I, National Research Council of Canada, Atomic Energy Project Report MT 92.

Mark, J. C. 1945, The spherical harmonic method II, National Research Council of Canada, Atomic Energy Project Report MT 97.

Meghreblian, R. V. and Holmes, D. K. 1960, Reactor Analysis, McGraw-Hill Book Co., New York.

Moore, F. K. and Gibson, W. E. 1960, Propagation of weak disturbances in a gas subject to relaxation effects. IAS J. 27, 117-127.

Pearson, W. E. 1964, On the direct solution of the governing equation for a radiation-resisted shock waves. NASA TN D-2128.

Sobolev, V. V. 1963, A Treatise on Radiative Transfer, D. Van Nostrand Co., Inc., New York.

Somerfeld, A. 1964, Partial Differential Equations of Physics, Paperback Edition, Academic Press Inc., New York.

Traugott, S. C. 1963, A differential approximation for radiative transfer with application to normal shock structure. Procedings of the 1963 Heat Transfer and Fluid Mechanics Institute, Stanford University Press, Stanford.

Vincenti, W. G. 1959, Nonequilibrium flow over a wavy wall, J. Fluid Mech. 6, 481-496.

Vincenti, W. G., and Baldwin, B. S., Jr. 1962, Effect of thermal radiation on the propagation of plane acoustic waves, J. Fluid Mech. 12, 449-477.

Weinberg, A. M. and Wigner, E. P. 1958, The Physical Theory of Neutron Chain Reactors, The University of Chicago Press, Chicago.